Florian Cajori.

THE

ELEMENTS OF EUCLID;

CONTAINING THE

FIRST SIX, AND THE ELEVENTH AND TWELFTH BOOKS,

CHIEFLY

FROM THE TEXT OF DR. SIMSON;

ADAPTED TO

ELEMENTARY INSTRUCTION BY THE INTRODUCTION

OF

SYMBOLS.

BY

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PREFACE.

Although a thorough knowledge of the Elements of Euclid is indispensable previously to any further progress in the Mathematics; yet such is the repulsive form in which they have hitherto been presented to the student, that he seldom fails to experience considerable embarrassment in the onset, and frequently abandons the pursuit after reading the first four or five propositions.

The absence of methodical arrangement in any kind of argument, seldom fails to obscure the whole. Each syllogism should be clearly detached from the other, so that its force be fully evinced, before another, as consequent upon it, be brought into notice. In the present editions of Euclid no stronger mark of distinction exists between the steps of the demonstration than a colon or period. The student is therefore extremely liable to blend them together. And the nice discrimination necessary to separate them, requires more labour and greater abstraction of thought than the generality of beginners are either capable of, or willing to submit to. They are in great danger of hurrying from one step to the other, without clearly comprehending the meaning of any; until arriving at the conclusion, instead of perceiving a demonstration, they have acquired only a confused idea of letters and angles.

In this is comprised the chief part, if not the whole, of the difficulty experienced at the threshold of the science; and which, it is hoped, the present work will effectually remove.

The editor claims to himself no more originality of thought than the application only, to a novel purpose, of a system already in use, though to a limited extent, in the University of Cambridge. It has however undergone some considerable, but essential alterations, in order to render it available in elementary instruction.

The plan is simply this;—the appropriation of a single line or paragraph to every individual step throughout the proposition. This will exhibit the whole train of argument in a perspicuous and methodical arrangement. In order also to facilitate the object in view, by making the sentences shorter and more concise, symbols are substituted for words of frequent occurrence. Considerable attention has been devoted to the selection of these. All that appeared to be mere arbitrary characters have been rejected; while those only are retained whose figure or property makes them appropriate emblems of that which they are intended to indicate. As soon, therefore, as the eye has become familiarized with them, the sense will be much easier perceived, than if the ideas were expressed at length in alphabetical characters.

The text of Dr. R. Simson forms the basis of the work. Wherever he has been deviated from, recourse has been had in every case to the judgment of certain individuals whose acknowledged scientific learning rendered their advice decisive. By these gentlemen also, the editor has been influenced in the choice of the symbols; and materially assisted in other respects.

It was originally intended to supply algebraical demonstrations to the second and fifth books. This has however been relinquished, under the apprehension that the size, and consequently the expense of the work, would be so increased, as to hazard the probability of its introduction into schools.*

It is necessary to make some apology for relinquishing the symbol for the phrase "is similar to." It has indeed been adopted by Mr. Barlow in his "Theory of Numbers;" where it occurs so often as to render it extremely serviceable. Its use in the present publication may not be so manifest, and during the progress of the work it was deemed adviseable, by more competent judges than the editor, not to continue it. It will be found to occur, however, in not more than one or two instances, where, from the sheets having been struck off, it was too late to make the alteration.

Queen's College, *May* 21, 1827.

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* "A new translation of the Elements, &c." by Mr. George Phillips, embraces all that is requisite on this point.

were one in the judgment of [illegible]

It was originally intended to [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] This has been [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

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EXPLANATION OF THE SYMBOLS.

+ signifies *plus*, or *together with.*
− - - *minus*, or *less by.*
× - - *into.*
÷ - - *is divided* or *divided by.*
= - - *is equal to.*
≠ - - *is unequal to.*
> - - *is greater than.*
≯ - - *is not greater than.*
< - - *is less than.*
≮ - - *is not less than.*
⊥ - - *is perpendicular to.*
∥ - - *is parallel to.*
∦ - - *is not parallel to.*
∵ - - *because.*
∴ - - *therefore.*
AB or $\overline{AB}$ is a *right line* terminated by the points A and B.
∠ - - *angle.*
∠s - - *angles.*
△ - - *triangle.*
▱ - - *parallelogram.*
Sol. ▱ - *parallelopiped.*
⊙ - - *circle.*
○ - - *circumference.*
½⊙ - *semicircle.*
$\overset{\frown}{AB}$ - *arc*, terminated by the points A and B.
$\overline{AB}^2$ - *square described on the right line AB.*
$\overline{AB+CD}^2$ - *square* described on the whole right line made up of the two AB and CD.

AB × CD is a *rectangle* contained by the right lines AB and CD.

A : B signifies *the ratio of A to B.*

A : B : : C : D *the ratio of A to B is the same as the ratio of C to D;* and is thus read:— *as A is to B so is C to D;* or, *A is to B as C to D.*

Dupl. of A : B *the duplicate ratio of A to B.*

Tripl. of A : B *the triplicate ratio of A to B.*

ABREVIATIONS.

Alti.	is short for	*altitude.*
Alter.	- - -	*alternate.*
Bis.	- - -	*bisect.*
Circumscr.	- -	*circumscribe.*
Coin.	- - -	*coincide.*
Com.	- - -	*common.*
Constr.	- - -	*construct.*
Cont.	- - -	*contain.*
Descr.	- - -	*describe.*
Diagr.	- - -	*diagram.*
Diag.	- - -	*diagonal.*
Dist.	- - -	*distance.*
Divis.	- - -	*divisions.*
Ea.	- - - -	*each.*
Ex.	- - -	*exterior.*
Homol.	- - -	*homologous.*
Hxgn.	- - -	*hexagon.*
In. int.	- - -	*interior.*
Mag.	- - -	*magnitude.*
No.	- - -	*number.*
Opp.	- - -	*opposite.*
Pl.	- - - -	*plane.*
Plygn.	- - -	*polygon.*
Prod.	- - -	*produce.*
Pt.	- - - -	*point.*
Ptgn.	- - -	*pentagon.*
Pyr.	- - -	*pyramid.*
Rem.	- - -	*remainder.*
Rt.	- - - -	*right.*

Sec.	is short for	*section.*
Sect.	- - -	*sector.*
Seg.	- - - -	*segment.*
Simil.	- - -	*is similar to.*
Sol.	- - - -	*solid.*
Sph.	- - -	*sphere.*
Sq.	- - - -	*square.*
Ver.	- - - -	*vertical.*
Whl.	- - -	*whole.*

THE

ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I.

A point is that which has no parts, or which has no magnitude.

II.

A line is length without breadth.

III.

The extremities of lines are points.

IV.

A right line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of superficies are lines.

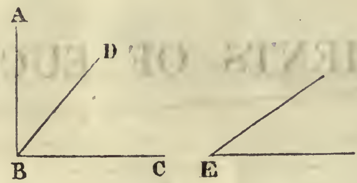
VII.

A plane superficies is that in which any two points being taken, the right line between them lies wholly in that superficies, VIII.

"A plane angle is the inclination of two lines to each other in a plane which meet together, but are not in the same right line."

IX.

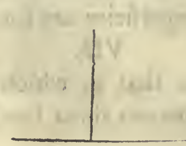
A plane rectilineal angle is the inclination of two right lines to one another, which meet together, but are not in the same right line.



'N.B. When several angles are at one point B, either of 'them is expressed by three letters, of which the letter that 'is at the vertex of the angle, that is, at the point in which 'the right lines that contain the angle meet one another, 'is put between the other two letters, and one of these two is 'somewhere upon one of these right lines, and the other 'upon the other line. Thus the angle which is contained by the 'right lines AB, CB, is named the angle ABC, or CBA; 'that which is contained by AB, DB, is named the angle 'ABD, or DBA; and that which is contained by DB, CB, 'is called the angle DBC, or CBD. But, if there be only 'one angle at a point, it may be expressed by the letter at 'that point; as the angle at E.'

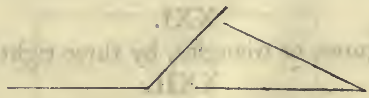
X.

When a right line standing on another right line makes the adjacent angles equal to each other, each of these angles is called a right angle; and the right line which stands on the other is called a perpendicular to it.



XI.

An obtuse angle is that which is greater than a right angle.



XII.

An acute angle is that which is less than a right angle.

XIII.

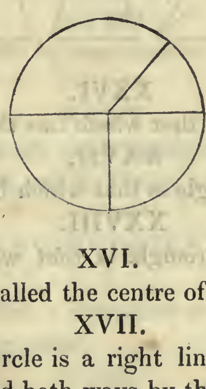
" A term or boundary is the extremity of any thing."

XIV.

A figure is that which is inclosed by one or more boundaries.

XV.

A circle is a plain figure contained by one line, which is called the circumference, and is such that all right lines drawn from a certain point within the figure to the circumference, are equal to one another.



XVI.

And this point is called the centre of the circle.

XVII.

A diameter of a circle is a right line drawn through the centre, and terminated both ways by the circumference.

XVIII.

A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

" A segment of a circle is the figure contained by a right line and that part of the circumference it cuts off."

XX.

Rectilineal figures are those which are contained by right lines.

XXI.

Trilateral figures, or triangles, by three right lines.

XXII.

Quadrilateral, by four right lines.

XXIII.

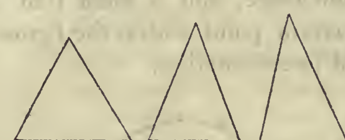
Multilateral figures, or polygons, by more than four right lines.

XXIV.

Of three sided figures, an equilateral triangle is that which has three equal sides.

XXV.

An isosceles triangle is that which has only two sides equal.



XXVI.

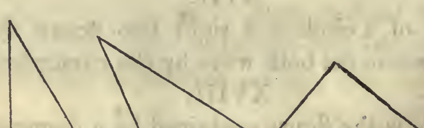
A scalene triangle is that which has three unequal sides.

XXVII.

A right angled triangle is that which has a right angle.

XXVIII.

An obtuse angled triangle is that which has an obtuse angle.

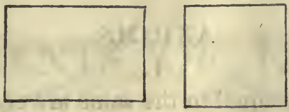


XXIX.

An acute angled triangle is that which has three acute angles.

XXX.

Of quadrilateral or four sided figures, a square has all its sides equal and all its angles right angles.

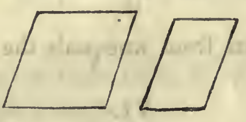


XXXI.

An oblong has all its angles right angles, but has not all its sides equal.

XXXII.

A rhombus has all its sides equal, but its angles are not right angles.



XXXIII.

A rhomboid has its opposite sides equal to each other, but all its sides are not equal, nor its angles right angles.

XXXIV.

All other four sided figures besides these are called trapeziums.

XXXV.

Parallel right lines are such as are in the same plane, and which, being produced ever so far do not meet.*

POSTULATES.

I.

Let it be granted that a right line may be drawn from any one point to any other point.

II.

That a terminated right line may be produced to any length in a right line.

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* To these may be added:—

1. A problem is a proposition denoting something to be done.
2. A theorem is a proposition which requires to be demonstrated.
3. A corollary is a consequent truth gained from a preceding demonstration.
4. A deduction is a proposition drawn from a preceding demonstration

III.

And that a circle may be described from any centre at any distance from that centre.

AXIOMS.

I.

Things which are equal to the same are equal to each other.

II.

If equals be added to equals the wholes are equal.

III.

If equals be taken from equals the remainders are equal.

IV.

If equals be added to unequals the wholes are unequal.

V.

If equals be taken from unequals the remainders are unequal.

VI.

Things which are double of the same are equal to each other.

VII.

Things which are halves of the same are equal to each other.

VIII.

Magnitudes which coincide with each other, that is, which exactly fill the same space, are equal to each other.

IX.

The whole is greater than its part.

X.

Two right lines cannot enclose a space.

XI.

All right angles are equal to each other.

XII.

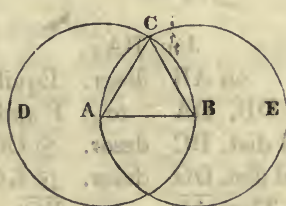
"If a right line meet two right lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these right lines being continually produced shall at length meet on that side on which are the angles which are less than two right angles."

ELEMENTS OF EUCLID.

PROP. I.—Problem.

To describe an equilateral triangle upon a given finite right line.

Let AB be the given right line; it is required to describe on AB an equilateral triangle.



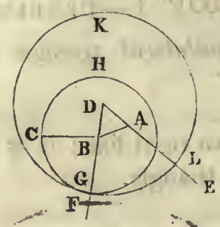
With cent. A, and dist. AB, descr. ⊙ BCD, 3 post.
with cent. B, and dist. BA, descr. ⊙ ACE;
and from C, draw CA, CB to A and B: 1 post.
Then ABC is an equilat. △.
For ∵ A is cent. ⊙ BCD,
∴ AC = AB; 15 definition.
and ∵ B is cent. ⊙ ACE,
∴ BC = BA.
But AC = AB,
∴ AC = BC; 1 axiom.
∴ AB, BC, CA = each other.

Wherefore △ ABC is equilat: and is described on AB. Q. E. F.

PROP. II.—Problem.

From a given point, to draw a right line equal to a given right line.

Let A be the given point, and BC the given right line; it is required to draw from A a right line = BC.



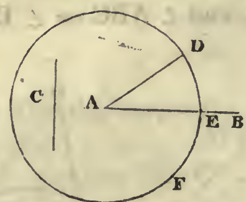
Join BA ;			1 post.
on AB descr.		Equilat. △ ABD,	I. I.
prod. DB, DA	to	F and E ;	2 post.
with cent. B, and dist. BC	descr.	⊙ CGH,	3 post.
and with cent. D, and dist. DG	descr.	⊙ KGL.	
Then $\overline{AL}$	=	$\overline{BC}$.	
For ∵ pt. B is cent. ⊙ CGH,			
∴ BC	=	BG ;	15 def.
and ∵ D is cent. ⊙ KGL,			
∴ DL	=	DG,	
but part $\overline{DA}$	=	part $\overline{DB}$,	constr.
∴ rem. AL	=	rem. BG :	3 ax.
but BC	=	BG ;	
∴ $\overline{AL}$	=	$\overline{BC}$.	1 ax.

Wherefore from A has been drawn $\overline{AL}$ = $\overline{BC}$. Q. E. F.

PROP. III.—Problem.

From the greater of two given right lines to cut off a part equal to the less.

Let C and AB be the given right lines, of which AB > C; it is required to cut off from AB a part = C.



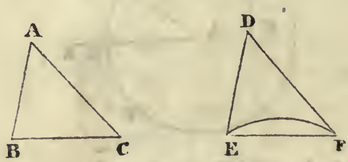
From A draw AD	=	C;	2. 1.
with cent. A and dist. AD	descr.	⊙ DEF;	
so that it cut AB	in	E:	
then AE	=	C.	
For ∵ A is cent. ⊙ DEF,			
∴ AE	=	AD;	15 def.
But C	=	AD;	constr.
∴ AE	=	C.	1 ax.

Wherefore from the greater AB is cut off AE = C the less.
Q. E. F.

PROP. IV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to each other; they shall likewise have their bases, or third sides, *equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, viz. those to which the equal sides are opposite.*

Let the two $\triangle$s ABC, DEF have AB = DE and AC = DF; also the $\angle$ BAC = $\angle$ EDF. Then base BC = base EF; and $\triangle$ ABC = $\triangle$ DEF; and $\angle$ ABC = $\angle$ DEF; and $\angle$ BCA = $\angle$ EFD.



For if $\triangle$ ABC	be applied	to $\triangle$ DEF,	
so that pt. A	be on	pt. D,	
and AB	on	DE;	
then, $\because$ AB	=	DE,	hyp.
$\therefore$ B	coincides with	E;	
and $\because$ $\angle$ BAC	=	$\angle$ EDF,	hyp.
$\therefore$ AC	coin. with	DF;	
and $\because$ AC	=	DF,	hyp.
$\therefore$ C	coin. with	F:	
But B	coin. with	E,	
$\therefore$ BC	coin. with	EF.	

For if BC does not coin. with EF,
then two right lines enclose a space, — 10 ax.
which is impossible.

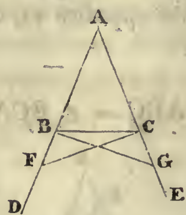
$\therefore$ BC	coin. with and =	EF;	8 ax.
also $\triangle$ ABC	coin. with and =	$\triangle$ DEF;	
and $\angle$ ABC	coin. with and =	$\angle$ DEF;	
and $\angle$ BCA	coin. with and =	$\angle$ EFD.	

Wherefore if two triangles, &c. &c. Q. E. D.

PROP. V.—Theorem.

The angles at the base of an isosceles triangle are equal to each other; and if the equal sides be produced, the angles on the other side of the base shall be equal.

Let ABC be an isosceles △, and let AB, AC be prod. to D and E; then ∠ ABC = ∠ BCA and ∠ DBC = ∠ BCE.



In AD take any pt. F;
make AG = AF; 3. 1.
and join BG, CF.
∵ AF = AG, constr.
and AB = AC, hyp.
and that ∠ FAG is com. to △s AFC, AGB;
∴ BG = CF,
also ∠ ABG = ∠ ACF, 4. 1.
and ∠ AFC = ∠ AGB.
Again, ∵ whole AF = whole AG,
and part AB = part AC;
∴ rem. BF = rem. CG : 3 ax.
and ∵ BG = CF,
and BF = CG,
and that ∠ BFC = ∠ CGB;
∴ ∠ BCF = ∠ CBG,
and ∠ BCG = ∠ CBF; 4. 1.
which are ∠s on opp. side base BC.
Again ∵ ∠ ABG = ∠ ACF,
and ∠ BCF = ∠ CBG;
∴ rem. ∠ ABC = rem. ∠ BCA. 3 ax.
which are ∠s at base BC.

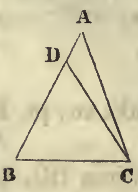
Wherefore the angles, &c. &c. Q. E. D.

Cor. Hence every equilateral triangle is also equiangular.

PROP. VI.—Theorem.

If two angles of a triangle be equal to each other, the sides also which subtend, or are opposite to, *the equal angles, shall be equal to one another.*

Let $\triangle$ ABC have $\angle$ ABC $=$ $\angle$ BCA; then AB $=$ AC.



For if AB $\neq$ AC;
One of them is $>$ the other:
let AB $>$ AC;
and cut off DB $=$ AC. 3. 1.
Join DC.
Then $\because$ DB $=$ AC,
and BC is com. to $\triangle$s DBC, ACB,
and that $\because$ $\angle$ DBC $=$ $\angle$ BCA; hyp.
$\therefore$ AB $=$ DC,
and $\triangle$ DBC $=$ ACB, } 4. 1.
i. e. the less $=$ greater,
which is absurd.
$\therefore$ AB not $\neq$ AC,
i. e. AB $=$ AC.

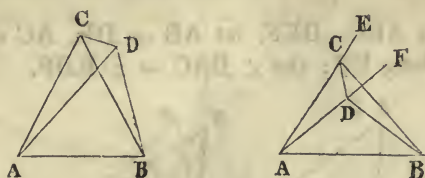
Wherefore if two angles, &c. &c. Q. E. D.

Cor. Hence every equiangular triangle is also equilateral.

PROP. VII.—Theorem.

Upon the same base and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to each other, and likewise those which are terminated in the other extremity.

If possible on same base AB and on the same side, let the two △s ACB, ADB have CA of one = DA of the other, both which are terminated in pt. A of the base; and likewise CB = DB which are terminated in B.



Join CD

First—Let ea. of the vertices of the △s fall without the other △.

∵ AC	=	AD,	hyp.
∴ ∠ ACD	=	∠ ADC;	5. 1.
but ∠ ACD	>	∠ BCD,	9 ax.
∴ ∠ ADC	>	∠ BCD,	
much more ∴ ∠ BDC	>	∠ BCD.	
Again, ∵ BD	=	BC,	hyp.
∴ ∠ BDC	=	∠ BCD,	5. 1.
but also ∠ BDC	>	∠ BCD;	demon.

which is absurd.

Secondly—Let vertex D of △ ADB fall within the other △ ACB.

prod. AC, AD to E and F.

Then ∵ AC	=	AD,	hyp.
∴ ∠ ECD	=	∠ CDF;	5. 1.
but ∠ ECD	>	BCD,	9 ax.
∴ ∠ CDF	>	BCD,	
much more ∴ ∠ BDC	>	BCD.	
Again ∵ BD	=	BC,	hyp.
∴ ∠ BDC	=	BCD,	5. 1.
but also ∠ BDC	>	BCD.	demon.

which is absurd.

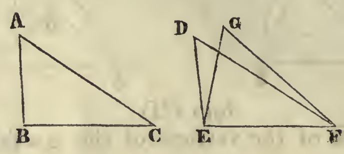
Thirdly—The case of the vertex of one △ being on a side of the other, needs no demonstration.

Wherefore upon the same, &c. &c. Q. E. D.

PROP. VIII.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; the angle which is contained by the two sides of the one shall be equal to the angles contained by the two sides equal to them, of the other.

Of the $\triangle$s ABC, DEF. let AB = DE, AC = DF, and base BC = base EF; the $\angle$ BAC = $\angle$ EDF.



For if $\triangle$ ABC	be appl. to	$\triangle$ DEF,	
so that pt. B	be on	E,	
and BC	on	EF;	
then $\because$ BC	=	EF,	hyp.
$\therefore$ shall pt. C	coin. with	F;	
and $\because$ BC	coin. with	EF,	
$\therefore$ BA, AC	coin. with	ED, DF.	
For, if BA, AC	do not coin. with	ED, DF;	
let BA, AC	coin. with	EG, GF:	

Then upon same base EF are constituted two $\triangle$s in a manner which has been demonstrated to be impossible. 7.1.

$\therefore$ if BC	coin. with	EF,	
BA, AC	must coin. with	ED, DF,	
and $\therefore$ $\angle$ BAC	coin. with	$\angle$ EDF;	
$\therefore$ $\angle$ BAC	=	$\angle$ EDF.	8 ax.

Wherefore if two triangles, &c. &c. Q. E. D.*

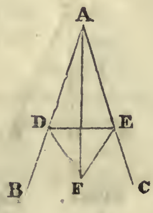
---

* Dr. Barrow, in his edition of the Elements, deduces from this Proposition and the fourth.—I. that "triangles mutually equilateral are also mutually equiangular," and II. that "triangles mutually equilateral are equal to each other."

PROP. IX.—Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

Let ∠ BAC be the given rectilin. ∠ ; it is required to bisect it.



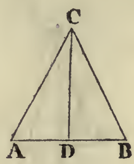
In AB	take	any pt. D ;	
make AE	=	AD ;	3. 1.
	Join DE.		
On DE	descr.	Equilat. △ DEF ;	1. 1.
	Join AF ;		
then rectilin. ∠ BAC	is bis. by	AF.	
∵ AE	=	AD	constr.
and AF	is com. to	△s DAF, EAF	
and base DF	=	base EF	constr.
∴ ∠ DAF	=	∠ EAF.	8. 1.

Wherefore rectilin. ∠ BAC is bisected by AF. Q. E. F.

PROP. X.—PROBLEM.

To bisect a given finite right line, that is, to divide it into two equal parts.

Let AB be the given right line; it is required to bisect AB.



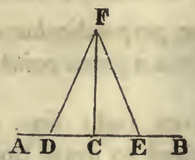
On AB	descr.	Equilat. △ ABC;	1. 1.
Bisect ∠ ACB	by	CD.	9. 1.
Then AB	is bis. in	D.	
∵ AC	=	CB,	constr.
and CD	is com. to	△s ACD, BCD,	
and ∠ ACD	=	∠ BCD;	constr.
∴ AD	=	DB.	4. 1.

Wherefore $\overline{AB}$ is bisected in D. Q. E. F.

PROP. XI.—Problem.

To draw a right line at right angles to a given right line, from a given point in the same.

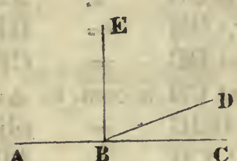
Let AB be the given right line and C the given point in it; it is required to draw a right line from the point C at right ∠s to AB.



In AC take any pt. D;
and make CE = CD. 3. 1.
On DE desc. Equilat. △ DEF; 1. 1.
Join FC;
Then FC is drawn at right ∠s to AB.
∵ CD = CE,
and FD = FE, constr.
and that FC is com. to △s DFC, EFC;
∴ ∠ DCF = ∠ FCE; 8. 1.
∴ ea. of these ∠s is a rt. ∠; 10 def. i.
∴ FC is at rt. ∠s to AB.

Wherefore from the point C in AB, FC has been drawn at right ∠s to AB. Q. E. F.

Cor. By help of this problem, it may be demonstrated, that two right lines cannot have a common segment.



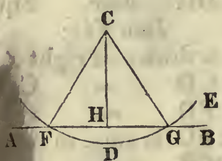
If it be possible,
let the segment AB be com. to two rt. lines ABC, ABD:
from B draw BE at rt. ∠s to AB:
and ∵ ABC is a right line,
∴ ∠ CBE = ∠ EBA. 10 def.
Similarly ∵ ABD is a right line,
∴ ∠ DBE = ∠ EBA;
and ∴ ∠ DBE = ∠ CBE; 1 ax.
i. e. less = greater.
which is absurd.

Therefore two right lines cannot have a common segment.

PROP. XII.—Problem.

To draw a right line perpendicular to a given right line of an unlimited length, from a given point without it.

Let AB be the given right line, and C the given point without it. It is required to draw from C a right line ⊥ to AB.



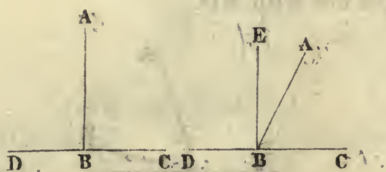
Take any pt. D on the other side of AB;			
With cent. C and dist. CD	desc.	⊙ FDGE;	15 def.
Bisect FG	in	H;	10. 1.
Join CF, CH and CG:			
Then is CH	⊥	AB.	
For ∵ GH	=	HF,	by constr.
and GC	=	CF,	15 def.
and that CH is com. to		△s FHC, GHC;	
∴ adj. ∠ GHC	=	adj. ∠ FHC;	8. 1.
and ∴ CH	⊥	AB.	10 def.

Wherefore, from the given pt. C, has been drawn CH ⊥ AB. Q. E. F.

PROP. XIII.—Theorem.

The angles which one right line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let AB make with CD, on same side of it, the ∠s DBA, ABC; these are either two right ∠s, or are together = two right ∠s.



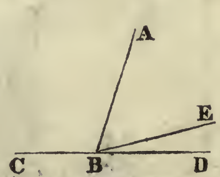
For if ∠ DBA = ∠ ABC,
then each is a right ∠. — 10 def.
But if ∠ DBA ≠ ∠ ABC,
from B draw BE rt. ∠s to DC; — 11. 1.
∴ right ∠ CBE = right ∠ EBD. — 11 ax.
And ∵ ∠ CBE = ∠ CBA + ∠ ABE,
add the ∠ EBD,
∴ ∠ CBE + ∠ EBD = ∠ CBA + ∠ ABE + ∠ EBD. — 2 ax.
Again, ∵ ∠ DBA = ∠ DBE + ∠ EBA,
add the ∠ ABC,
∴ ∠s DBA + ABC = ∠s DBE + EBA + ABC; — 2 ax.
but ∠ CBE + ∠ EBD = the same three ∠s;
∴ ∠ CBE + ∠ EBD = ∠s DBA + ABC. — 1 ax.
But ∠ CBE + ∠ EBD are two right ∠s,
∴ ∠ DBA + ∠ ABC = two right ∠s. — 1 ax.

Wherefore when a right line, &c. &c. Q. E. D.

PROP. XIV.—Theorem.

If, at a point in a right line two other right lines, upon the opposite side of it, make the adjacent angles together equal to two right angles, these two right lines shall be in one and the same right line.

At B in $\overline{AB}$ let $\overline{BC}$, $\overline{BD}$ on the opp. sides of $\overline{AB}$, make adj. $\angle$s ABC + ABD = 2 right $\angle$s. Then shall $\overline{CB}$ be in the same right line with $\overline{BD}$.



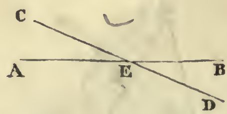
For if BD be not in same right line with BC,
Let BE be in same right line with BC.
Then, ∵ AB stands on CBE,
∴ $\angle$s ABE + ABC = 2 right $\angle$s; 13. 1.
but $\angle$s ABC + ABD = 2 right $\angle$s; by hyp.
∴ $\angle$s ABE + ABC = $\angle$s ABC + ABD;
remove com. $\angle$ ABC,
and ∴ rem. $\angle$ ABE = rem. $\angle$ ABD,
i. e. less = greater.
which is absurd.
Therefore BE is not in same right line with BC
And similarly none other than BD is in same right line with BC.

Wherefore, if at a point, &c. &c. Q. E. D.

PROP. XV.—Theorem.

If two right lines cut each other, the vertical or opposite angles shall be equal.

Let $\overline{AB}$, $\overline{CD}$ cut each other in E. The $\angle$ AEC $= \angle$ BED and $\angle$ AED $= \angle$ BEC.



$\because$ $\overline{AE}$ stands on		$\overline{CD}$,	
$\therefore$ $\angle$s AEC + AED	$=$	2 right $\angle$s.	13. 1.
Again, $\because$ $\overline{DE}$ stands on		$\overline{AB}$,	
$\therefore$ $\angle$s AED + DEB	$=$	2 right $\angle$s;	
$\therefore$ $\angle$s AEC + AED	$=$	$\angle$s AED + DEB;	1 ax.
remove	com.	$\angle$ AED,	
and rem. $\angle$ AEC	$=$	rem. $\angle$ DEB.	3 ax.
Similarly $\angle$ AED	$=$	$\angle$ BEC.	

Wherefore if two right lines cut each other, &c. &c. Q. E. D.

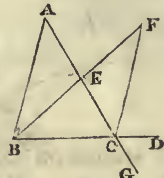
Cor. 1. From this it is manifest, that if two right lines cut each other, the angles they make at the point where they cut, are together equal to four right angles.

Cor. 2. And consequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP. XVI.—Theorem.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let the side BC of the △ ABC be prod. to D. Then ex. ∠ ACD > ABC or CAB.



Bisect AC	in	E;	10. 1.
Join BE;			
produce BE	to	F;	
make EF	=	EB;	3. 1.
Join FC;			
and prod. AC	to	G.	
Then ∵ AE	=	EC,	constr.
and BE	=	EF,	
and that ∠ AEB	=	∠ CEF;	15. 1.
∴ base AB	=	base FC,	4. 1.
and ∠ BAE	=	ECF;	
but ∠ ECD	>	∠ ECF,	9 ax.
∴ ∠ ACD	>	∠ BAE.	

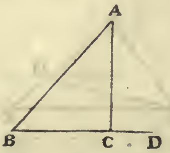
Similarly by bisecting BC, it may be demon: that ∠ BCG i.e. ∠ ACD > ∠ ABC.

Wherefore if one side, &c. &c. Q. E. D.

PROP. XVII.—Theorem.

Any two angles of a triangle are together less than two right angles.

Let ABC be any △, any two of its ∠s are together less than two right ∠s.



Prod. BC to D.

And ∵ ex. ∠ DCA > int. ∠ CBA, 16. 1.

add the ∠ ACB,

∴ ∠s DCA + ACB > ∠s CBA + ACB. 4 ax.

But ∠s DCA + ACB = 2 right ∠s; 13. 1.

∴ ∠s CBA + ACB < 2 right ∠s.

Similarly { ∠s BAC + ACB < 2 right ∠s,
and ∠s CAB + ABC < 2 right ∠s. }

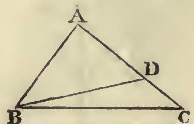
Wherefore, any two angles of a triangle, &c. &c. Q. E. D.

Cor. Hence in every triangle having a right or an obtuse angle, the other two angles are acute.

PROP. XVIII.—Theorem.

The greater side of every triangle subtends the greater angle.

Of △ ABC let side AC > side AB; then shall ∠ ABC be > ∠ ACB.



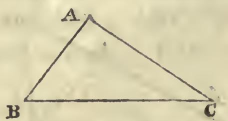
Since AC	>	AB,	
make AD	=	AB.	3. 1.
Join BD.			
∵ AD	=	AB,	constr.
∴ ∠ ABD	=	∠ ADB:	5. 1.
But ex. ∠ ADB	>	int. ∠ DCB;	16. 1.
∴ ∠ ABD	>	∠ ACB;	
much more ∴ ∠ ABC	>	∠ ACB.	

Wherefore the greater side of every triangle, &c. &c. Q. E. D.

PROP. XIX.—THEOREM.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Of $\triangle$ ABC let $\angle$ ABC be $>$ $\angle$ ACB; the side AC $>$ side AB.



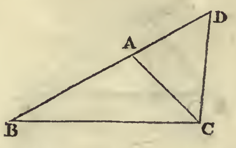
For if AC	$\ngtr$	AB,	
it is either $=$ or $<$ AB.			
FIRST—assume AC	$=$	AB,	
then $\angle$ ABC	$=$	$\angle$ ACB,	5. 1.
but $\angle$ ABC	$\neq$	$\angle$ ACB,	by hyp.
$\therefore$ AC	$\neq$	AB.	
SECONDLY—assume AC	$<$	AB;	
then $\angle$ ABC	$<$	$\angle$ ACB,	18. 1.
but $\angle$ ABC	$\nless$	$\angle$ ACB,	by hyp.
$\therefore$ AC	$\nless$	AB;	
and AC was demon.	$\neq$	AB;	
Therefore AC	$>$	AB.	

Wherefore the greater angle, &c. &c. Q. E. D.

PROP. XX.—Theorem.

Any two sides of a triangle are together greater than the third side.

Of $\triangle$ ABC, any two sides together, BA, AC $>$ BC, or AB, BC $>$ AC, or BC, CA $>$ AB.



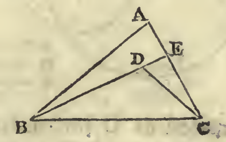
Prod. BA to D;
make AD $=$ AC;
Join DC.
Then $\because$ AD $=$ AC,
$\therefore \angle$ ADC $= \angle$ ACD; 5. 1.
but $\angle$ BCD $> \angle$ ACD, 9 ax.
$\therefore \angle$ BCD $> \angle$ ADC:
and $\because$ in $\triangle$ DCB; $\angle$ BCD $> \angle$ BDC,
$\therefore$ DB $>$ BC; 19. 1.
but DB $=$ BA$+$AC, by constr.
$\therefore$ sides BA$+$AC $>$ BC.
Similarly the sides $\left\{\begin{array}{l}\text{AB}+\text{BC} > \text{AC},\\ \text{BC}+\text{CA} > \text{AB}.\end{array}\right.$

Wherefore any two sides, &c. &c. Q. E. D.

PROP. XXI.—Problem.

If from the ends of a side of a triangle, there be drawn two right lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

From B and C, the ends of the side BC of $\triangle$ ABC, let BD, CD be drawn to pt. D within $\triangle$ ABC: then shall $BD + DC < BA + AC$, but shall contain $\angle BDC > \angle BAC$.



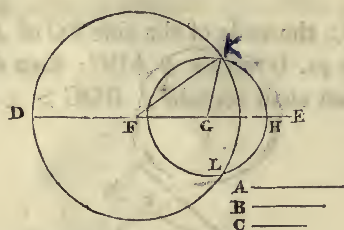
Prod. BD	to	E :	
$\because$ in $\triangle$ ABE; $BA + AE$	$>$	BE,	20. 1.
	add EC,		
$\therefore BA + AC$	$>$	$BE + EC$.	4 ax.
Again, $\because CE + ED$	$>$	CD,	20. 1.
	add DB,		
$\therefore CE + EB$	$>$	$CD + DB$;	4 ax.
but $BA + AC$	$>$	$BE + EC$,	
much more then $BA + AC$	$>$	$BD + DC$.	
Again, $\because$ in $\triangle$ CDE, ex. $\angle$ BDC	$>$	in. $\angle$ CED,	16. 1.
and that in $\triangle$ ABE, ex. $\angle$ CEB	$>$	in. $\angle$ BAC,	
$\therefore$ much more $\angle$ BDC	$>$	$\angle$ BAC.	

Wherefore, if from, &c. &c. Q. E. D.

PROP. XXII.—Problem.

To make a triangle having its sides equal to three given right lines, of which, any two whatever must be greater than the third.

Let A, B, C be the three given right lines of which A + B > C; A + C > B; and B + C > A: required to construct a △ having its sides = A, B, C respectively.



Take DE limited at D but unlim. towards E.

Cut off DF	=	A,	
——— FG	=	B,	3. 1.
and ——— GH	=	C;	
with cent. F and dist. FD	desc.	⊙ DKL,	
and with cent. G and dist. GH	desc.	⊙ HLK;	
∴ from K draw KF, KG	to	F and G:	
Then *sides* of △ KFG	=	A, B, and C ea. to ea.	
Because F	is cent.	⊙ DKL,	
∴ FK	=	FD;	15 def.
but FD	=	A,	by constr.
∴ FK	=	A.	1 ax.
Again, because G	is cent.	⊙ LKH,	
∴ GH	=	GK;	15 def.
but GH	=	C,	constr.
∴ GK	=	C:	1 ax.
and FG	=	B;	constr.

∴ the △ KFG

has its sides FK, KG, GF = rt. lines A, C, B ea. to ea.

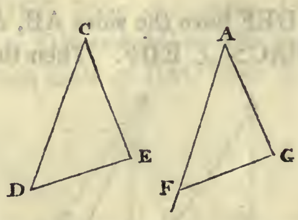
∴ △ KFG is drawn as required.

Q. E. F.

PROP. XXIII.—Problem.

At a given point in a given right line to construct a rectilineal angle equal to a given rectilineal angle.

Let A be the given point in the given right line AF, also ECD the given rectil. ∠; required to make an ∠ at pt. A in AF = rectil. ∠ DCE.



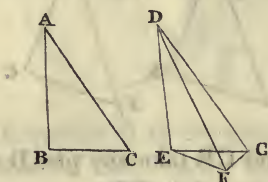
In CD and CE take any pts. D and E.
Join ED;
Constr. a △ AFG,
having AF, FG, GA = CD, DE, EC ea. to ea. 22. I.
∵ DC, CE = FA, AG ea. to ea.
and base ED = base GF
∴ ∠ GAF = ∠ ECD. 8. I.

Wherefore at given point A, in given right line AF, has been constr. a rectil. ∠ GAF = given rectil. ∠ ECD. Q. E. F.

PROP. XXIV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle, shall be greater than the base of the other.

Let △s ABC, DEF have the sides AB, AC = DE, DF ea. to ea. but the ∠ BAC > ∠ EDF. Then the base BC > EF.



Of the two sides DE, DF,

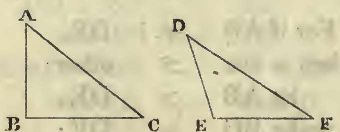
let DE	≯	DF.	
At D, in DE make ∠ EDG	=	∠ BAC;	23. 1.
make DG	=	AC or DF;	3. 1.
Join EG, GF.			
∵ AB	=	DE,	hyp.
and AC	=	DG,	constr.
and ∠ BAC	=	∠ EDG;	constr.
∴ base BC	=	base EG.	4. 1.
And ∵ DG	=	DF,	constr.
∴ ∠ DFG	=	∠ DGF;	5. 1.
but ∠ DGF	>	∠ EGF,	9 ax.
∴ ∠ DFG	>	∠ EGF;	
∴ much more ∠ EFG	>	∠ EGF:	
∴ EG	>	EF	19. 1.
but EG	=	BC,	
∴ BC	>	EF.	

Wherefore if two triangles, &c. &c. Q. E. D.

PROP. XXV.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other; the angle contained by the sides of the one which has the greater base, shall be greater than the angle contained by the sides, equal to them, of the other.

Let △s ABC, DEF, have the sides AB, AC = sides DE, DF, viz. AB = DE and AC = DF, but have the base BC > base EF; then shall ∠ BAC be > ∠ EDF.



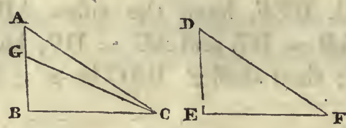
For if ∠ BAC	$\not>$	∠ EDF;	
it must be either	= or <	∠ EDF.	
First—assume ∠ BAC	=	EDF;	
then base BC	=	base EF;	4. 1.
but BC	$\neq$	EF,	hyp.
∴ ∠ BAC	$\neq$	∠ EDF.	
Secondly—assume ∠ BAC	<	∠ EDF;	
then BC	<	EF;	24. 1.
but BC	$\not<$	EF,	hyp.
∴ ∠ BAC	$\not<$	∠ EDF;	
and it was demon. that ∠ BAC	$\neq$	∠ EDF;	
∴ ∠ BAC	>	∠ EDF.	

Wherefore if two triangles, &c. &c. Q. E. D.

PROP. XXVI.—Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side; viz. either the sides adjacent to equal angles in each, or the sides opposite to them; then shall the other sides be equal, each to each, and also the third angle of the one equal to the third angle of the other.

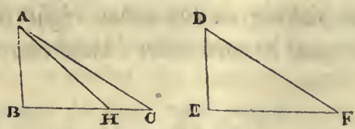
Let △s ABC, DEF, have ∠s ABC, BCA = ∠s DEF, EFD ea. to ea., viz. ∠ ABC = ∠ DEF and ∠ BCA = ∠ EFD; also one side equal to one side. First, let the adjacent side in ea. viz. BC = EF: then shall AB = DE and AC = DF, also ∠ BAC = ∠ EDF.



For if AB	≠	DE,	
then is one	>	other;	
let AB	>	DE;	
make BG	=	DE;	3. 1.
join GC.			
Then ∵ BG	=	DE,	
and BC	=	EF,	hyp.
and that ∠ GBC	=	∠ DEF,	hyp.
∴ base GC	=	base DF,	
and △ GBC	=	△ DEF,	4. 1.
and also ∠ GCB	=	∠ DFE;	
but ∠ DFE	=	∠ BCA,	hyp.
∴ ∠ BCG	=	∠ BCA,	1 ax.
i. e. less	=	greater;	
which is absurd.			
∴ AB	not ≠	DE,	
i. e. AB	=	DE:	
and ∵ AB	=	DE,	
and BC	=	EF,	hyp.
and that ∠ ABC	=	∠ DEF;	hyp.
∴ base AC	=	base DF,	4. 1.
and ∠ BAC	=	∠ EDF.	

PROP. XXVI. CONTINUED.

SECONDLY—let the sides opposite to equal $\angle$s in ea. $\triangle$, be equal to ea. other; viz. $AB = DE$; then shall $AC = DF$, $BC = EF$ and $\angle\ BAC = \angle\ EDF$.



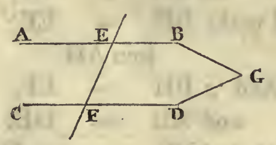
For if BC	$\neq$	EF,	
let BC	$>$	EF,	
and make BH	$=$	EF;	3. 1.
join AH.			
And $\because$ BH	$=$	EF,	
and AB	$=$	DE,	hyp.
and that $\angle$ ABH	$=$	$\angle$ DEF,	hyp.
$\therefore$ base AH	$=$	base DF,	4. 1.
and $\triangle$ ABH	$=$	$\triangle$ DEF,	4. 1.
and $\angle$ BHA	$=$	$\angle$ EFD;	4. 1.
but $\angle$ EFD	$=$	$\angle$ BCA,	hyp.
$\therefore$ $\angle$ BHA	$=$	$\angle$ BCA,	1 ax.
i. e. ex. $\angle$ BHA	$=$	in. and opp. $\angle$ BCA,	
which is impossible.			16. 1.
$\therefore$ BC not	$\neq$	EF,	
i. e. BC	$=$	EF.	
And $\because$ BC	$=$	EF,	
and AB	$=$	DE,	hyp.
and that $\angle$ ABC	$=$	$\angle$ DEF,	hyp.
$\therefore$ AC	$=$	DF,	4. 1.
and $\angle$ BAC	$=$	$\angle$ EDF.	4. 1.

Wherefore if two triangles, &c. Q. E. D.

PROP. XXVII.—Theorem.

If a right line falling on two other right lines, makes the alternate angles equal to each other; these two right lines shall be parallel.

Let EF falling on AB, CD, make alt. ∠ AEF = alt. ∠ EFD, then shall AB ∥ CD.



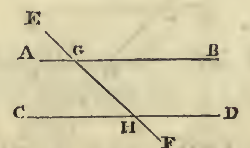
For, if AB ∦ CD,
they will meet, either towards A and C, or B and D;
produce AB and CD to meet in G, towards B and D;
then EGF is a △,
∴ ex. ∠ AEF > int. ∠ EFD; 16. 1.
but ∠ AEF = ∠ EFD, by hyp.
which is impossible;
∴ AB and CD do not meet towards B and D.
Similarly AB, CD do not meet towards A and C;
∴ AB ∥ CD. 35 def.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. XXVIII.—Theorem.

If a right line falling upon two other right lines, makes the exterior angle equal to the interior and opposite upon the same side of the line; or makes the interior angles upon the same side together equal to two right angles; the two right lines shall be parallel to each other.

Let EF falling on AB, CD make ex. ∠ EGB = in ∠ GHD. And also the ∠s BGH + GHD = two rt. ∠s. then shall AB ∥ CD.



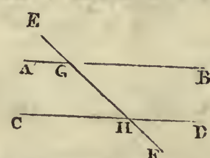
∵ ∠ EGB	=	∠ GHD,	hyp.
and ∠ AGH	=	∠ EGB,	15. 1.
∴ ∠ AGH	=	∠ GHD;	1 ax.
and these are altern. ∠s,			
∴ AB	∥	CD.	27. 1.
Again, ∵ ∠s BGH + GHD	=	2 rt. ∠s,	hyp.
and ∠s AGH + BGH	=	2 rt. ∠s,	13. 1.
∴ ∠s AGH + BGH	=	∠s BGH + GHD;	
take away com. ∠ BGH,			
∴ rem. ∠ AGH	=	rem. ∠ GHD;	
which are altern. ∠s,			
∴ AB	∥	CD.	27. 1.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. XXIX.—Theorem.

If a right line fall on two parallel right lines, it makes the alternate angles equal to each other; and the exterior angle equal to the interior and opposite angle upon the same side; and likewise the two interior angles on the same side together equal to two right angles.

Let EF fall on the parallels AB, CD; then shall alt. $\angle$ AGH = alt. $\angle$ GHD; and ex. $\angle$ EGB = in. $\angle$ GHD; also int. $\angle$s BGH + GHD = two right $\angle$s.



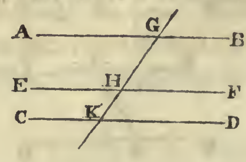
For if $\angle$ AGH	$\neq$	$\angle$ GHD,	
let $\angle$ AGH	$>$	$\angle$ GHD:	
then $\because$ $\angle$ AGH	$>$	$\angle$ GHD,	
	add $\angle$ BGH,		
$\therefore$ $\angle$ AGH + $\angle$ BGH	$>$	$\angle$ BGH + GHD;	4 ax.
but $\angle$s AGH + BGH	=	2 rt. $\angle$s,	13. 1.
$\therefore$ $\angle$s BGH + GHD	$<$	2 rt. $\angle$s,	
$\therefore$ AB, CD would meet if prod. far enough;			12 ax.
	but they do not meet		
for AB	$\parallel$	CD,	hyp.
$\therefore$ $\angle$ AGH	not $\neq$	$\angle$ GHD,	
i.e. $\angle$ AGH	=	$\angle$ GHD;	
but $\angle$ AGH	=	$\angle$ EGB,	15. 1.
$\therefore$ $\angle$ EGB	=	$\angle$ GHD;	1 ax.
	add $\angle$ BGH,		
$\therefore$ $\angle$ EGB + $\angle$ BGH	=	$\angle$ BGH + $\angle$ GHD;	2 ax.
but $\angle$s EGB + BGH	=	2 rt. $\angle$s,	13. 1.
$\therefore$ $\angle$s BGH + GHD	=	2 rt. $\angle$s.	1 ax.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. XXX.—Theorem.

Right lines which are parallel to the same right line are parallel to each other.

Let AB, CD be ea. ∥ EF; then shall AB ∥ CD.



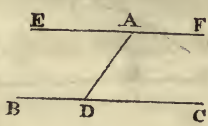
Let GK cut AB, EF, CD.
And ∵ GK falls on ∥ s AB, EF,
∴ alt. ∠ AGH = alt. ∠ GHF. 29. 1.
Again, ∵ GK falls on ∥ s EF, CD,
∴ ex. ∠ GHF = int. ∠ GKD; 29. 1.
but ∠ AGH = ∠ GHF,
∴ ∠ AGK = ∠ GKD; 1 ax.
and they are altern. ∠ s,
∴ AB ∥ CD. 27. 1.

Wherefore right lines, &c. &c. Q. E. D.

PROP. XXXI.—Problem.

To draw a right line through a given point, parallel to a given right line.

Let A be the given point, and BC the given right line; required to draw through A a right line $\parallel$ BC.



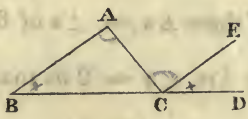
In BC take any pt. D;
join AD;
at A, in AD make $\angle$ DAE = $\angle$ ADC; 23. 1.
and prod. EA to F:
then shall EF $\parallel$ BC.
$\because$ AD falls on the rt. lines BC, EF,
and makes alt. $\angle$ EAD = alt. $\angle$ ADC,
$\therefore$ EF $\parallel$ BC.

Therefore through the given point A has been drawn a right line EAF $\parallel$ the given right line BC. Q. E. F.

PROP. XXXII.—Theorem.

If the side of a triangle be produced, the exterior angle is equal to the two interior and opposite angles: and the three interior angles of every triangle are together equal to two right angles.

Let side BC of △ ABC be prod. to D. The exterior ∠ ACD = two inter. opp. ∠s CAB + ABC; and the three interior ∠s ABC, BCA, CAB together = 2 rt. ∠s.



Through C draw CE ∥ BA. 31. 1.

∵ AC falls on ∥s BA, CE,

∴ alt. ∠ BAC = alt. ∠ ACE.

Again, ∵ BD falls on ∥s BA, CE,

∴ ex. ∠ ECD = int. & opp. ∠ ABC; 29.1.

but ∠ ACE = ∠ BAC,

∴ whole ex. ∠ ACD = 2int. ∠sCAB + ABC. 2 ax.

add ∠ ACB,

∴ ∠ ACD + ∠ ACB = ∠sCAB + ABC + ACB; 2 ax.

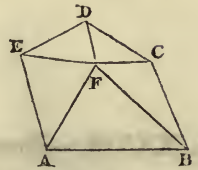
but ∠s ACD + ACB = 2 rt. ∠s, 13. 1.

∴ also ∠sABC + BCA + CAB = 2 rt. ∠s.

Wherefore if a side, &c. &c. Q. E. D.

Cor. 1. All the interior angles of any rectilineal figure are, together with four right angles, equal to twice as many right angles as the figure has sides.

For,



For, by drawing right lines from any point F within it to each of its angles, any rectil. Fig. ABCDE, may be divided into as many △s as there are sides to the figure. Then by the preceding proposition,

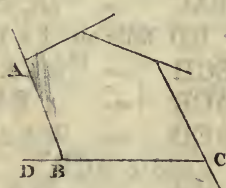
all the ∠s of these △s = 2 as many rt. ∠s as there are △s, i. e. sides to the fig.

and the same ∠s of these △s = ∠s of fig. + ∠s at pt. F, the common vertex;

i. e. all the ∠s of these △s = ∠s of fig. + 4 rt. ∠s, [2 cor. 15. 1.

∴ ∠s of fig. + 4 rt. ∠s = 2 as many rt. ∠s as the fig. has sides. 1 ax.

Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.



∵ Every int. ∠ ABC + its ex. ∠ ABD = 2 rt. ∠s, 13. 1.

∴ all int. ∠s + all ext. ∠s of the fig. = 2 as many rt. ∠s as the fig. has sides;

i. e. all int. ∠s + all ext. ∠s of fig. = all int. ∠s + 4 rt. ∠s;

remove the interior ∠s which are common,

∴ all. ex. ∠s = 4 rt. ∠s.

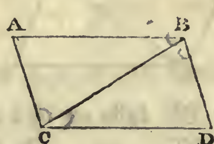
[Hence by this proposition it is manifest that if the angle contained by the equal sides of an isosceles triangle be a right angle, then the other two angles must be each half a right angle.

And also that the angles of an equilateral triangle are each equal to two thirds of a right angle.]

PROP. XXXIII.—Theorem.

The right lines which join the extremities of two equal and parallel right lines towards the same parts, are also themselves equal and parallel.

Let AB, CD be equal and parallel right lines, and joined towards the same parts by the right lines AC, BD; AC and BD are also equal and parallel.



Join BC;

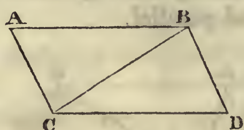
∵ BC	falls on	∥ s AB, CD,	
∴ alt. ∠ ABC	=	alt. ∠ BCD:	29. 1.
and ∵ AB	=	CD,	hyp.
and BC	com. to	△ s ABC, BCD,	
and ∠ ABC	$\neq$	∠ BCD,	
∴ AC	=	BD,	4. 1.
and △ ABC	=	△ BCD,	
and ∠ ACB	=	∠ CBD:	
and ∵ BC	falls on	AC, BD,	
and makes alt. ∠ ACB	=	alt. ∠ CBD,	27. 1.
∴ AC	∥	BD;	
and also AC	=	BD.	demon.

Wherefore the right lines, &c. &c. Q. E. D.

PROP. XXXIV.—Theorem.

The opposite sides and angles of parallelograms are equal to each other, and the diameter bisects them, that is, divides them into two equal parts.

Let AD be a * ▫, and let BC be its diam. Then AB = CD, AC = BD; also ∠ ABD = ∠ DCA and ∠ CAB = ∠ BDC. Also diam. BC bis. ▫ AD.



∵ BC falls on ∥ s AB, CD,

∴ alt. ∠ ABC = alt. ∠ BCD. 29. 1.

Similarly, ∵ AC ∥ BD,

∴ ∠ ACB = ∠ CBD;

∴ In the △ s ABC, BCD,

the ∠ s ABC, BCA = ∠ s BCD, CBD ea. to ea.

and BC is com.

∴ AB = CD,
AC = BD, } 26. 1.
and ∠ CAB = ∠ BDC.

And, ∵ ∠ ABC = ∠ BCD,

and ∠ CBD = ∠ ACB,

∴ whole ∠ ABD = whole ∠ DCA; 2 ax.

and it was demon. ∠ CAB = ∠ BDC.

Again, ∵ AB = CD,

and BC is com.

and that ∠ ABC = ∠ BCD,

∴ △ ABC = △ BCD; 4. 1.

∴ diam. BC bis. ▫ AD.

Wherefore the opp. &c. &c. Q. E. D.

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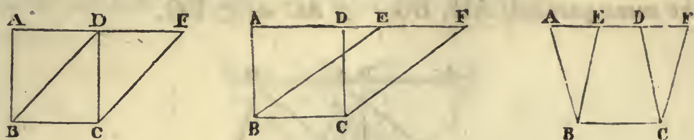
* For the sake of brevity, the diagonal letters only of parallelograms are expressed.

PROP. XXXV.—Theorem.

Parallelograms upon the same base and between the same parallels, are equal to each other.

Let ▭s ABCD, EBCF be on same base BC and between same parallels AF, BC. The ▭ AC = ▭ EC.



If AD, DF, opp. to BC, be term. in D,

then ea. ▭ AC, DC	=	2 △ BDC,	34. 1.
and ∴ ▭ AC	=	▭ DC.	6 ax.

But if AD, EF opp. to BC be not term. in D;

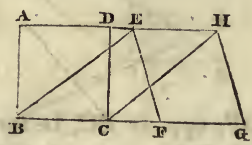
Then, ∵ AC	is a	▭,	
∴ AD	=	BC; }	34. 1.
Similarly EF	=	BC; }	
∴ AD	=	EF;	1 ax.
and DE is com.			
∴ whole or rem. AE	=	whole or rem. DF:	
and ∵ AE	=	DF,	
and AB	=	DC,	34. 1.
and that ex. ∠ FDC	=	in. ∠ EAB,	29. 1.
∴ EB	=	FC, }	4. 1.
and △ EAB	=	△ FDC; }	
from trape. ABCF	take	△ FDC,	
and also from the same	take	△ EAB,	
and rem.	=	rem.	3 ax.
i. e. ▭ AC	=	▭ EC.	

Therefore parallelograms, &c. &c. Q. E. D.

PROP. XXXVI.—THEOREM.

Parallelograms on equal bases and between the same parallels are equal to each other.

Let □ AC, EG be upon equal bases BC, FG, and between the same parallels AH, BG. □ AC = □ EG.



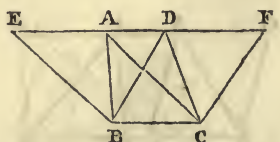
Join BE, CH.				
∵ BC	=	FG,		hyp.
and FG	=	EH,		34. 1.
∴ BC	=	EH;		1 ax.
∴ EC	is a	□:		33. 1.
and □ EC	=	□ AC		
for they are on same base BC, &c.				35. 1.
Similarly □ EC	=	□ EG;		
∴ □ AC	=	□ EG.		1 ax.

Wherefore parallelograms on equal bases, &c. &c. Q. E. D.

PROP. XXXVII.—THEOREM.

Triangles on the same base and between the same parallels are equal to each other.

Let $\triangle$s ABC, DBC be on same base BC and between same parallels AD, BC. $\triangle$ ABC = $\triangle$ DBC.



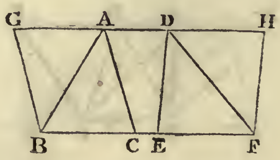
Prod. AD both ways	to	E and F ;	
through B draw BE	$\parallel$	CA ;	31. 1.
and through C draw CF	$\parallel$	BD ;	
$\therefore$ ea. fig. EC, FB	is a	$\square$:	34 def.
and $\because$ they are on same base BC, &c.			
$\therefore$ $\square$ EC	=	$\square$ FB ;	35. 1.
and $\because$ diam. AB	bis.	$\square$ EC,	
$\therefore$ $\triangle$ ABC	=	$\frac{1}{2}$ $\square$ EC ;	34. 1.
similarly $\triangle$ DBC	=	$\frac{1}{2}$ $\square$ FB ;	
$\therefore$ $\triangle$ ABC	=	$\triangle$ DBC.	7 ax.

Wherefore triangles, &c. &c. Q. E. D.

PROP. XXXVIII.—Theorem.

Triangles upon equal bases and between the same parallels are equal to each other.

Let △s ABC, DEF be on equal bases BC, EF, and between same parallels AD, BF. Then △ ABC = △ DEF.



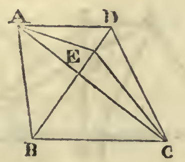
Produce AD both ways	to	G and H;	
through B draw BG	∥	CA;	31. 1.
and through F draw FH	∥	ED;	
then each fig. GC, HE	is a	□;	34 def. 1.
and ∵ they are on equal bases, BC, EF, &c.			36. 1.
∴ □ GC	=	□ HE:	
and ∵ diam. AB	bis.	□ GC,	
∴ △ ABC	=	½ □ GC;	34. 1.
similarly △ DEF	=	½ □ HE;	
∴ △ ABC	=	△ DEF.	7 ax.

Wherefore triangles on equal bases, &c. &c. Q. E. D.

PROP. XXXIX.—Theorem.

Equal triangles upon the same base and on the same side of it, are between the same parallels.

Let the equal $\triangle$s ABC, DBC be on the same base BC and upon the same side of it; they are between the same parallels.



Join AD :

then AD $\parallel$ BC :

for, if AD $\nparallel$ BC :

through A draw AE $\parallel$ BC ; 31. 1.

and join EC ;

then $\triangle$ ABC $=$ $\triangle$ EBC ; 37. 1.

but $\triangle$ ABC $=$ $\triangle$ DBC, hyp.

$\therefore$ $\triangle$ DBC $=$ $\triangle$ EBC ; 1 ax.

i. e. greater $=$ less ;

which is impossible.

$\therefore$ AE $\nparallel$ BC ;

Similarly none but AD $\parallel$ BC ;

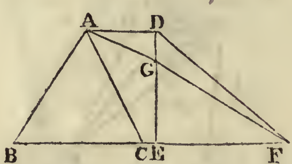
$\therefore$ AD $\parallel$ BC.

Wherefore equal triangles, &c. &c. Q. E. D.

PROP. XL.—Theorem.

Equal triangles upon equal bases in the same right line and towards the same parts, are between the same parallels.

Let the equal $\triangle$s ABC, DEF be on the equal bases BC, EF in same right line BF; and towards same parts; they are between same parallels.



Join AD:

then AD $\parallel$ BF :

for if AD $\nparallel$ BF,

through A draw AG $\parallel$ BF, 31. 1.

and join GF;

then $\triangle$ ABC $=$ $\triangle$ GEF, 38. 1.

but $\triangle$ ABC $=$ $\triangle$ DEF, hyp.

$\therefore$ $\triangle$ DEF $=$ GEF, 1 ax.

i. e. greater $=$ less;

which is impossible.

$\therefore$ AG $\nparallel$ BF.

Similarly none but AD $\parallel$ BF;

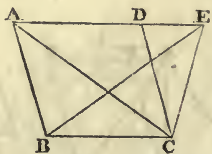
$\therefore$ AD $\parallel$ BF.

Wherefore equal triangles, &c. &c. Q. E. D.

PROP. XLI.—Theorem.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double of the triangle.

Let the ▱ BD and △ EBC be on the same base BC and between same parallels BC, AE; ▱ BD = 2 △ EBC.



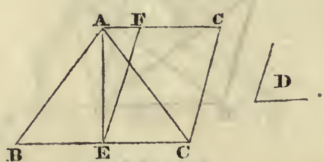
Join AC;
then △ ABC = △ EBC;
for they are on same base, &c. 37. 1.
And ∵ diam. AC bis. ▱ BD,
∴ ▱ BD = 2 △ ABC; 34. 1.
∴ also ▱ BD = 2 △ EBC.

Therefore if a parallelogram, &c. &c. Q. E. D.

PROP. XLII.—Problem.

To describe a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given △ and D the given rectilin. ∠. It is required to describe a ▱ = △ ABC and having an angle = ∠ D.



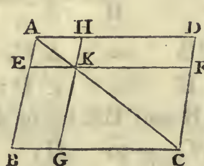
Bis. BC	in	E;	10. 1.
Join AE;			
at E in EC make ∠ CEF	=	∠ D;	23. 1.
Through A draw AFG	∥	BC;	31. 1.
through C draw CG	∥	EF;	31. 1.
∴ FC	is a	▱.	34 def. 1.
And ∵ base BE	=	base EC,	constr.
∴ △ ABE	=	△ ACE;	38. 1.
and ∴ the whl. △ ABC	=	2 △ ACE:	
but ▱ FC	=	2 △ ACE,	41. 1.
∴ ▱ FC	=	△ ABC;	6 ax.
and it has the ∠ CEF	=	∠ D, by constr.	

Wherefore a ▱ FECG has been constructed = △ ABC having an ∠ = ∠ D. Q. E. F.

PROP. XLIII.—THEOREM.

The Complements of the parallelograms which are about the diameter of any parallelogram, are equal to each other.

Let ABCD be a □, of which the diam. is AC; and EH, GF □s, about AC, and BK, KD the Complements. The Comp. BK = Comp. KD.



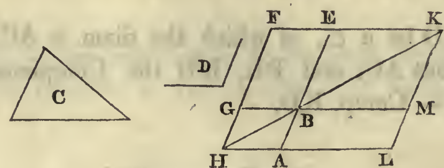
∵ Diam. AC	bis.	□ BD,		
∴ △ ABC	=	△ ACD;		
and similarly { △ AEK	=	△ AKH,	}	34. 1.
△ KGC	=	△ KCF;		
∴ △ AEK + △ KGC	=	△ AKH + △ KCF:		2 ax.
but whole △ ABC	=	whole △ ACD,		
∴ rem. Comp. BK	=	rem. Comp. KD.		3 ax.

Wherefore the Complements, &c. &c. Q. E. D.

PROP. XLIV.—Problem.

To a given right line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given rt. line, C the given △, and D the given rectil. ∠. Required to apply to AB a ▭ = △ C having an ∠ = ∠ D.



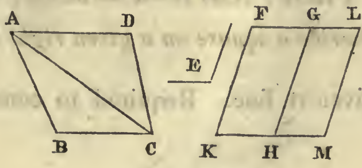
Make ▭ FB	=	△ C,	42. 1.
and having an ∠ at B	=	∠ D;	
and so, that AB and BE be in one rt. line;			
prod. FG	to	H;	
through A draw AH	‖	BG or EF;	31. 1.
join HB.			
Then, ∵ HF	falls on	‖ s AH, FE,	
∴ ∠ s AHF + HFE	=	2 rt. ∠ s;	29. 1.
∴ ∠ s BHF + HFE	<	2 rt. ∠ s;	
and ∴ will HB	meet	FE if prod. far enough;	12 ax.
let HB prod.	meet	FE prod. in K;	
through K draw KL	‖	EA, or FH;	31. 1.
and prod. HA, GB	to	L, M;	
then FL	is a	▭;	
and HK is diam. of ▭ FL;			
also AG, ME	are	▭ about HK;	
and LB, BF	=	Compls.	
∴ LB	=	BF;	43. 1.
but BF	=	△ C,	constr.
∴ LB	=	△ C:	1 ax.
and ∵ ∠ GBE	=	∠ ABM,	15. 1.
and also	=	∠ D,	constr.
∴ ∠ ABM	=	∠ D.	1 ax.

Therefore to the rt. line AB, the ▭ LB is applied = △ C, having the ∠ ABM = ∠ D. Q. E. F.

PROP. XLV.—Problem.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilin. fig. and E the given rectilin. ∠. Required to describe a ▱ = fig. BD and having an ∠ = ∠ E.



Join AC;

make ▱ FH	=	△ ADC,	42. 1.
having ∠ FKH	=	∠ E;	
to GH apply ▱ GM	=	△ ABC,	44. 1.
and having ∠ GHM	=	∠ E;	
the fig. FM	is the	▱ required.	
∵ ea. of ∠s, FKH, GHM	=	∠ E	constr.
∴ ∠ FKH	=	∠ GHM;	1 ax.
add ∠ KHG,			
∴ ∠ FKH + ∠ KHG	=	∠ KHG + ∠ GHM;	2 ax.
but ∠s FKH + KHG	=	2 rt. ∠s,	29. 1.
∴ also ∠s KHG + GHM	=	2 rt. ∠s;	1 ax.
and ∴ KH is in same rt. line with HM:			14. 1.
and ∵ GH	falls on	∥s KM, FG,	
∴ alt. ∠ MHG	=	alt. ∠ HGF;	29. 1.
add ∠ HGL,			
∴ ∠ MHG + ∠ HGL	=	∠ HGF + ∠ HGL;	2 ax.
but ∠s MHG + HGL	=	2 rt. ∠s,	29. 1.
∴ also ∠s HGF + HGL	=	2 rt. ∠s;	1 ax.
and ∴ FG is in same rt. line with GL:			14. 1.
and ∵ KF	∥	HG,	constr.
and HG	∥	ML,	
∴ KF	∥	ML;	30. 1.
and also KM	∥	FL,	constr.
∴ FM	is a	▱:	34 def. 1.
and ∵ △ ADC	=	▱ FH,	constr.
also △ ABC	=	▱ GM,	
∴ whole fig. BD	=	whole ▱ FM.	2 ax.

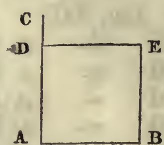
Wherefore ▱ FM has been described = rectil. fig. BD, and having ∠ FKM = ∠ E. Q. E. F.

Cor. From this it is manifest how to a given right line to apply a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure; viz. by applying to the given right line a parallelogram equal to the first triangle ABD and having an angle equal to a given angle.

PROP. XLVI.—Problem.

To describe a square on a given right line.

Let AB be given rt. line. Required to construct a square on AB.



From A draw AC rt. ∠s to AB;			11. 1.
make AD	=	AB;	3. 1.
through D draw DE	∥	AB;	31. 1.
and through B draw BE	∥	AD;	31. 1.
∴ fig. AE	is a	□;	34 def. 1.
and ∴ AB	=	DE,	34. 1.
and AD	=	BE,	
∴ □ AE	is	Equilat.	1 ax.
Again, ∵ AD	falls on	∥s AB, DE,	
∴ ∠s BAD + ADE	=	2 rt. ∠s;	29. 1.
but ∠ BAD	is a	rt. ∠,	constr.
∴ ∠ ADE	is a	rt. ∠;	
∴ opp. ∠s to these, are rt. ∠s,			
i. e. ea. of ∠s ABE, BED	is a	rt. ∠;	34. 1.
∴ □ AE	is	rectang.	1 ax.
also □ AE	is	Equilat.	
∴ □ AE	is a	sq.	30 def.

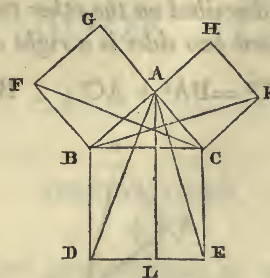
Wherefore a square ABDE has been described on given rt. line AB. Q. E. F.

Cor. Hence every parallelogram which has one right angle has all its angles right angles.

PROP. XLVII.—Theorem.*

In any right-angled triangle, the square which is described upon the side subtending the right angle, is equal to the squares described upon the sides which contain the right angle.

Let the right-angled △ ABC have the rt. ∠ BAC. Then $BC^2 = BA^2 + AC^2$.



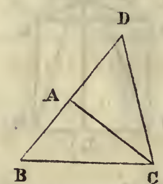
On BC descr. sq. BE;
on BA descr. sq. BG; } 46. 1.
and on AC descr. sq. AK;
draw AL ∥ BD or CE; 31. 1
Join AD, FC;
∵ ∠s BAC + BAG = two rt. ∠s, hyp. and 30 def.
∴ GA is in same rt. line with AC. 14. 1.
Similarly AB is in same rt. line with AH.
And ∵ ∠ DBC = ∠ FBA, 11 ax.
add to ea. ∠ ABC,
∴ whole ∠ DBA = whole ∠ FBC: 2 ax.
and ∵ AB, BD = FB, BC ea. to ea. 30 def.
and ∠ DBA = ∠ FBC,
∴ △ ABD = △ CBF. 4. 1.
Now ▭ BL = 2 △ ABD,
also sq. GB = 2 △ CBF, } 41.1.
(for they are respectively on same bases, &c.)
∴ sq. GB = ▭ BL: 6 ax.
Similarly, by joining AE and BK, it may be dem.
that sq. AK = ▭ CL;
∴ sqs. GB + AK = whole sq. BE 2 ax.
but sqs. GB, AK, BE were descr. on $\overline{AB}$, $\overline{AC}$, $\overline{BC}$, respectively,
∴ $BC^2 = BA^2 + AC^2$.
Wherefore the square of the side, &c. &c. Q. E. D.

* This proposition has been demonstrated several ways:—vide Clavius, Schouler, Ashby, Leslie, &c. &c.; but of all these, this, which is the original, is most generally admired for its simplicity and elegance.

PROP. XLVIII.—Theorem.

If a square described on one of the sides of a triangle, be equal to the squares described on the other two sides of it; the angle contained by these two sides is a right angle.

Of △ ABC let $BC^2 = BA^2 + AC^2$; ∠ BAC is a rt. ∠.



From A draw AD	rt. ∠s to	AC;	11. 1.
make AD	=	AB;	3. 1.
	join DC.		
Then, ∵ DA	=	AB,	
∴ DA^2	=	AB^2;	
	add AC^2,		
∴ $DA^2 + AC^2$	=	$AB^2 + AC^2$;	2 ax.
but DC^2	=	$DA^2 + AC^2$,	47. 1.
(for DAC	is	rt. ∠),	constr.
also BC^2	=	$BA^2 + AC^2$,	hyp.
∴ DC^2	=	BC^2;	1 ax.
and ∴ DC	=	BC;	
and ∵ DA	=	AB,	constr.
and AC	is com. to	△s DAC, BAC,	
and also DC	=	BC,	
∴ ∠ DAC	=	∠ BAC;	8. 1.
but ∠ DAC	is a	rt. ∠,	constr.
∴ ∠ BAC	is a	rt. ∠.	

Therefore if a square, &c. &c. Q. E. D.

BOOK II.

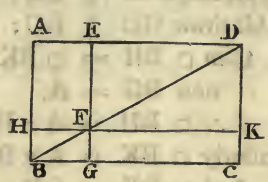
DEFINITIONS.

I.

Every right angled parallelogram, *or rectangle,* is said to be contained by any two of the right lines which contain one of the right angles.*

II.

In every parallelogram, any of the parallelograms about the diameter, together with the two complements, is called a Gnomon. " Thus the □ HG + complements AF, FC, is the " gnomon, which is more briefly expressed by the letters " AGK, or EHC, which are at the opposite angles of the " parallelograms which make the gnomon."

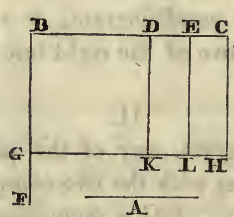


* The opposite sides of parallelograms, and consequently rectangles, being equal; it is evident that the product of any two of the adjacent sides, i. e. of those which contain a right angle, will be the area or content of the whole. And thus for the sake of brevity, a rectangle is said to be contained as in the definition. And which is expressed by connecting the adjacent sides by sign (×) of multiplication, thus the right angled parallelogram AC is called AB × AD, which is thus read " the rectangle AB, AD."

PROP. I.—Theorem.

If there be two right lines, one of which is divided into any number of parts; the rectangle contained by the two right lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be the two right lines; and let BC be divided into any number of parts in D and E; then A × BC = A × BD, A × DE, A × EC.



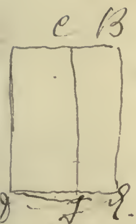
From B, draw BF at rt. ∠s to BC; 11. 1.
make BG = A;
&thro. D, E, C draw DK, EL, & CH ∥ BG; } 31. 1.
through G. draw GH ∥ BC; }
then □ BH = □ BK + □ DL + □ EH;
now BG = A, constr.
∴ □ BH is A × BC.
Similarly □ BK is A × BD.
And ∵ DK = GB, 34. 1.
and GB = A,
∴ DK = A; 1 ax.
and ∴ □ DL is A × DE.
Similarly □ EH is A × EC;
∴ A × BC = A × BD, A × DE, A × EC, together.

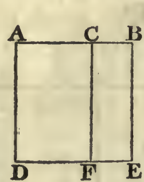
Wherefore if two right lines, &c. &c. Q. E. D.

PROP. II.—THEOREM.

*If a right line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.**

Let $\overline{AB}$ be divided into any two parts in C; then $AB \times BC + AB \times AC = AB^2$.





On AB	desc.	sq. AE;	46. 1.
thro. C draw CF	‖	AD or BE.	31. 1.
Then ∵ DA	=	AB,	30 def. 1.
∴ □ AF	is	$AB \times AC$.	
Again, ∵ BE	=	AB,	30 def. 1.
∴ □ CE	is	$AB \times BC$;	
but □ AF + □ CE	=	whole □ AE;	
and AE	is	AB^2;	constr.
∴ $AB \times BC + AB \times AC$	=	AB^2.	

Wherefore if a right line, &c. &c. Q. E. D.

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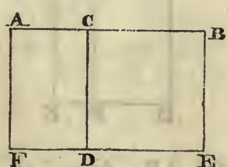
* A similar demonstration will apply should the right line be divided into any number of parts.

PROP. III.—THEOREM.

If a right line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let $\overline{AB}$ be divided into any two parts in C; then $AB \times BC = AC \times CB + CB^2$.



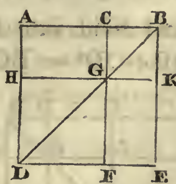
On BC	desc.	sq. CE;	46. 1.
prod. ED	to	F;	
thro. A draw AF	$\parallel$	CD or BE.	31. 1.
Then, $\because$ CD	=	CB,	30 def. 1.
$\therefore \square$ AD	is	$AC \times CB$;	
and by constr. $\square$ DB	is	CB^2;	
but $\square$ DB + $\square$ AD	=	whole $\square$ AE.	
And $\because$ BE	=	BC,	30 def. 1.
$\therefore \square$ AE	is	$AB \times BC$:	
$\therefore AB \times BC$	=	$AC \times CB + CB^2$.	1 ax.

Therefore if a right line be divided, &c. &c. Q. E. D.

PROP. IV.—THEOREM.

If a right line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let $\overline{AB}$ be divided into any two parts in C: Then $AB^2 = AC^2 + CB^2 + 2\,AC \times CB$.



On AB descr.		sq. AE;	46. 1.
thro. C, draw CF	∥	BE or AD;	31. 1.
and thro. G, draw HK	∥	AB or DE.	31. 1.
Then, ∵ BD meets		∥ s AD, CF,	
∴ ex. ∠ CGB	=	int. ∠ ADB;	29. 1.
but ∠ ADB	=	∠ ABD,	5. 1.
(for AD	=	AB,)	30 def. 1.
∴ ∠ CGB	=	∠ CBG;	1 ax.
and ∴ also BC	=	CG;	6. 1.
but BC	=	GK,	34. 1.
and CG	=	BK,	34. 1.
∴ □ CK	is	equilat.	1 ax. 1.
Again, ∵ CB meets		∥ s CG, BK,	
∴ ∠ s KBC + BCG	=	2 rt. ∠ s;	29. 1.
but ∠ KBC	is a	rt. ∠,	30 def. 1.
∴ ∠ BCG	is a	rt. ∠;	1 ax.
and ∴ □ CK	is	rectang.	1 ax.
wherefore □ CK	is a	sq. i. e. CB^2.	
Similarly HF	is a	sq. i. e. AC^2,	
(for HG	=	AC).	34. 1.
And ∵ compl. AG	=	compl. GE,	43. 1.
and □ AG	is	$AC \times CB$,	
(for GC	=	CB),	30 def. 1.
∴ □ GE	=	$AC \times CB$,	1 ax.
and ∴ □ AG + □ GE	=	$2\,AC \times CB$:	
and □ s HF, CK.	are	AC^2, CB^2,	
∴ □ s HF, CK, AG, GE together	=	$AC^2 + CB^2 + 2\,AC \times CB$;	
but □ s HF, CK, AG, GE	=	whole □ AE,	
and □ AE	is	AB^2,	
∴ AB^2	=	$AC^2 + CB^2 + 2\,AC \times CB$.	

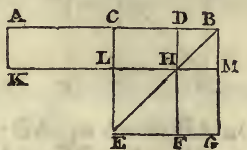
Wherefore if a right line, &c. &c. Q. E. D.

Cor. From the demonstration, it is manifest that the parallelograms about the diameter of a square are likewise squares.

PROP. V.—Theorem.

If a right line be divided into two equal parts and also into two unequal parts; the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

Let $\overline{AB}$ be bis. in C and divid. into two unequal parts in D. Then shall $AD \times DB + CD^2 = BC^2$.



On BC	descr.	sq. CG;	46. 1.
	join BE;		
thro. D, draw DF	∥	BG or CE;	31. 1.
thro. H, draw KM	∥	CB or EG;	
and thro. A, draw AK	∥	CL or BM.	
∵ compl. CH	=	compl. HG,	43. 1.
	add □ DM,		
∴ whole □ CM	=	whole □ DG;	2 ax.
but □ CM	=	□ AL,	36. 1.
(for AC	=	CB),	hyp.
∴ □ AL	=	□ DG;	
	add CH,		
and ∴ whole □ AH	=	gnom. CMF:	2 ax.
but □ AH	=	$AD \times DB$,	
(for DH	=	DB,)	30 def. 1. and cor. 4. 2.
∴ gnom. CMF	=	$AD \times DB$;	1 ax.
add □ LF	=	CD^2,	cor. 4. 2. and 34. 1.
∴ gnom. CMF + LF	=	$AD \times DB + CD^2$;	2 ax.
but CMF + LF	=	fig. CG,	
and □ CG	is	BC^2,	constr.
∴ $AD \times DB + CD^2$	=	BC^2.	

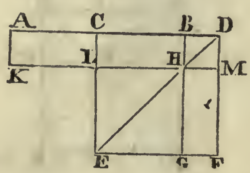
Wherefore if a right line, &c. &c. Q. E. D.

From this it is manifest, that the difference of the squares of two unequal lines AC, CD, is equal to the rectangle contained by their sum and difference.

PROP. VI.—Theorem.

If a right line be bisected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the right line which is made up of the half and the part produced.

Let $\overline{AB}$ be bis. in C and prod. to D; $AD \times DB + BC^2 = CD^2$.



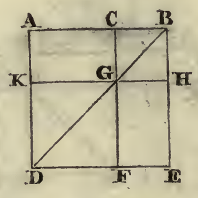
On CD	descr.	sq. CF;	46. 1.
	join DE;		
thro. D, draw DF	∥	BG or CE;	31. 1.
thro. H, draw KM	∥	CD or EF;	31. 1.
and thro. A, draw AK	∥	CL or BH.	31. 1.
∵ compl. CH	=	compl. HF,	43. 1.
and that □ AL	=	□ CH,	36. 1.
(for AC	=	CB,)	hyp.
∴ □ AL	=	□ HF;	
	add CM,		
and ∴ whole □ AM	=	gnom. CMG:	2 ax.
but □ AM	=	$AD \times DB$,	
(for DM	=	DB),	cor. 4. 2; 34 def. 1.
∴ gnom. CMG	=	$AD \times DB$:	1 ax.
add □ LG	=	CB^2,	cor. 4. 2; 34. 1.
∴ gnom. CMG + □ LG	=	$AD \times DB + CB^2$;	2 ax.
but CMG + LG	=	□ CF i. e. CD^2,	
∴ $AD \times DB + CB^2$	=	CD^2.	1 ax.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. VII.—Theorem.

If a right line be divided into any two parts, the squares of the whole line and one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let $\overline{AB}$ be divid. into any two parts in C. Then $AB^2 + BC^2 = 2\ AB \times BC + AC^2$.



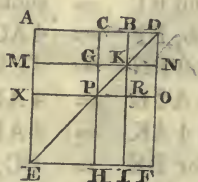
On AB descr. sq. AE;			46. 1.
and constr. the fig. as in the preceding.			
Then, ∵ compl. AG	$=$	compl. GE,	43. 1.
add $\square$ CH,			
∴ $\square$ AH	$=$	$\square$ CE;	2 ax.
∴ $\square$ AH + $\square$ CE	$=$	2 $\square$ AH:	
but $\square$ AH + $\square$ CE	are	gnom. AHF + sq. CH,	
∴ gnom. AHF + sq. CH	$=$	2 $\square$ AH;	1 ax.
but $2\ AB \times BC$	$=$	2 $\square$ AH,	
(for BH	$=$	BC),	cor. 4. 2. and 30 def. 1.
∴ gnom. AHF + sq. CH	$=$	$2\ AB \times BC$;	1 ax.
add $\square$ KF	$=$	AC^2,	cor. 4. 2. and 34. 1.
∴ gnom. AHF + sq. CH + sq. KF	$=$	$2\ AB \times BC + AC^2$;	2 ax.
but AHF + CH + KF	$=$	whole fig. AE + CH,	
and AE + CH	$=$	$AB^2 + BC^2$,	
∴ $AB^2 + BC^2$	$=$	$2\ AB \times BC + AC^2$.	1 ax.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. VIII.—Theorem.

If a right line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square of the other part, is equal to the square of the right line, which is made up of the whole and that part.

Let $\overline{AB}$ be divid. into any two parts in C. Then $4 . AB \times BC + AC^2 = \overline{AB + BC}^2$.*



Prod. AB	to	D;	
make BD	=	BC;	
on AD descr. sqr. AF;			
and construct 2 figs. as in the preceding.			
∵ CB	=	BD,	constr.
and CB	=	GK,	34. 1.
and that BD	=	KN,	34. 1.
∴ GK	=	KN:	1 ax.
similarly PR	=	RO.	
And ∵ CB	=	BD,	
and GK	=	KN,	
∴ □ CK	=	□ BN,	36. 1.
and □ GR	=	□ RN;	36. 1.
but □ CK	=	□ RN,	43. 1.
∴ □ BN	=	□ GR;	
∴ □ s BN, CK, GR, and RN	=	each other:	
∴ BN, CK, GR, and RN	=	4 CK.	
Again, ∵ CB	=	BD,	
and BD	=	BK, i. e. CG,	cor. 4. 2.

* $\overline{AB + BC}^2$, denotes the square described on the whole line which is made up of the two AB, BC.

And

PROP. VIII.—CONTINUED.

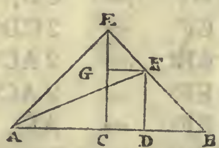
and that CB	=	GK, i.e. GP,	
∴ CG	=	GP:	
and ∵ CG	=	GP,	
and PR	=	RO,	
∴ $\square$ AG	=	$\square$ MP,	36. 1.
and $\square$ PL	=	$\square$ RF:	
but compl. MP	=	compl. PL,	43. 1.
∴ $\square$ AG	=	$\square$ RF;	
∴ $\square$s AG, PM, PL, and RF	=	each other;	
and ∴ AG, PM, PL, RF	=	4 AG;	
but BN, CK, GR, and RN	=	4 CK;	demon.
∴ gnom. AOH	=	4 AK;	
but 4 AK	=	$4 AB \times BC$,	
(for BK	=	BC,)	
∴ $4 AB \times BC$	=	gnom. AOH;	
add $\square$ XH	=	AC^2,	cor. 4. 2.
∴ $4 AB \times BC + AC^2$	=	gnom. AOH + $\square$ XH;	2 ax.
but whl. fig. AF	=	AOH + XH,	
and AF	=	AD^2,	
∴ $4 AB \times BC + AC^2$	=	AD^2;	
but AD^2	=	$\overline{AB + BC}^2$,	
∴ $4 AB \times BC + AC^2$	=	$\overline{AB + BC}^2$.	

Wherefore if a right line, &c. &c. Q. E. D.

PROP. IX.—Theorem.

If a right line be divided into two equal, and also into two unequal parts; the squares of the two unequal parts are together double of the square of half the line, and of the square of the line between the points of rection.

Let $\overline{AB}$ be divided into two unequal parts in D, and two = parts in C. $AD^2 + DB^2 = 2\,AC^2 + 2\,CD^2$.



From C draw CE at rt. ∠s to AB;
make CE = AC, or CB;
Join EA, EB;
thro. D, draw DF ∥ CE;
thro. F, draw FG ∥ AB;
Join AF.

Then, ∵ AC	=	CE,	
∴ ∠ EAC	=	∠ AEC;	5. 1.
but, ∵ ∠ ACE	is a	rt. ∠,	const.
∴ ea. of the ∠s EAC, AEC	=	½ rt. ∠.	
Similarly ea. of the ∠s CEB, EBC	=	½ rt. ∠;	
∴ whl. ∠ AEB	=	rt. ∠.	
And ∵ ∠ GEF	is	½ rt. ∠,	
and ∠ EGF	is a	rt. ∠,	29. 1.
(for ∠ EGF	= int. rt. ∠ ECB,)		29. 1.
∴ rem. ∠ EFG	=	½ rt. ∠;	
and ∴ ∠ GEF	=	∠ EFG;	1 ax.
and ∴ GE	=	FG.	6. 1.
Again, ∵ ∠ at B	is	½ rt. ∠,	

 and

PROP. IX.—CONTINUED.

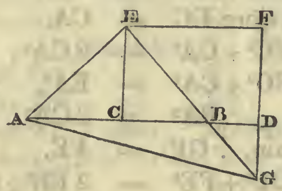
and $\angle$ FDB	is a	rt. $\angle$,	29.1.
(for $\angle$ FDB	=	int. and rt. $\angle$ ECB,)	
$\therefore$ rem. $\angle$ BFD	=	$\frac{1}{2}$ rt. $\angle$;	
and $\therefore$ $\angle$ B	=	$\angle$ BFD;	
and $\therefore$ DF	=	DB.	
And $\because$ AC	=	CE,	constr.
$\therefore$ AC^2	=	CE^2,	
$\therefore$ AC^2+CE^2	=	$2\,AC^2$;	2 ax.
but AC^2+CE^2	=	AE^2,	47. 1.
$\therefore$ AE^2	=	$2\,AC^2$.	
Similarly EF^2	=	$2\,GF^2$;	
but GF	=	CD,	34. 1.
$\therefore$ EF^2	=	$2\,CD^2$;	
and also AE^2	=	$2\,AC^2$,	demon.
$\therefore$ AE^2+EF^2	=	$2\,AC^2+2\,CD^2$;	
but AE^2+EF^2	=	AF^2,	47. 1.
(for $\angle$ AEF	is a	rt. $\angle$,)	
$\therefore$ AF^2	=	$2\,AC^2+2\,CD^2$;	
but AF^2	=	AD^2+DF^2,	
(for $\angle$ ADF	is a	rt. $\angle$,)	
$\therefore$ AD^2+DF^2	=	$2\,AC^2+2\,CD^2$;	
but DF	=	DB,	
$\therefore$ AD^2+DB^2	=	$2\,AC^2+2\,CD^2$.	

Wherefore if a right line, &c. Q. E. D.

PROP. X.—Theorem.

If a right line be bisected, and produced to any point, the square of the whole line thus produced, and the square of the part of it produced, are together double of the square of half the line bisected, and of the square of the line made up of the half and the part produced.

Let $\overline{AB}$ be divided into two = parts in C, and produced to D. Then $AD^2 + DB^2 = 2\ AC^2 + 2\ CD^2$.



From C draw CE at rt. ∠s to AB;
make CE = CA or CB;
Join AE, EB;
Thro. E, draw EF ∥ AB;
thro. D, draw DF ∥ CE;
∵ EF meets ∥s EC, FD,
∴ ∠s CEF + EFD = 2 rt. ∠s; 29. 1.
and ∴ ∠s BEF + EFD < 2 rt. ∠s;
∴ EB and FD will meet if prod. towards B and D; 12 ax.
prod. EB, FD to meet in G;
Join AG.
Then, ∵ AC = CE, constr.
∴ ∠ CEA = ∠ EAC; 5. 1.
but ∠ ACE is a rt. ∠, constr.
∴ ea. of the ∠s CEA, EAC = ½ rt. ∠; } 32.1.
Similarly ea. of the ∠s CEB, EBC = ½ rt. ∠; }
∴ ∠ AEB is a rt. ∠.

And,

PROP. X.—CONTINUED.

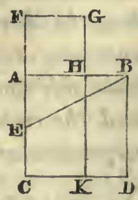
And, $\because \angle$ EBC	=	$\frac{1}{2}$ rt. $\angle$,	
$\therefore \angle$ DBG	=	$\frac{1}{2}$ rt. $\angle$;	15. 1.
and $\because$ alt. rt. $\angle$ ECD	=	alt. $\angle$ CDG,	29. 1.
$\therefore \angle$ BDG	is a	rt. $\angle$;	
and $\therefore$ rem. $\angle$ DGB	=	$\frac{1}{2}$ rt. $\angle$;	
and $\therefore \angle$ DGB	=	$\angle$ DBG;	
and $\therefore$ BD	=	DG.	6.1.
Again, $\because$ EG	meets	$\parallel$ s BD, EF,	
$\therefore$ ex. $\angle$ DBG	=	int. $\angle$ GEF;	29. 1.
but $\angle$ DBG	=	$\angle$ DGB,	
$\therefore \angle$ GEF	=	$\angle$ FGE;	
and $\therefore$ GF	=	FE.	6.1.
Now, since EC	=	CA,	constr.
$\therefore EC^2+CA^2$	=	$2\,CA^2$;	2 ax.
but EC^2+CA^2	=	EA^2,	47.1.
$\therefore EA^2$	=	$2\,CA^2$.	
Again, $\because$ GF	=	FE,	
$\therefore GF^2+FE^2$	=	$2\,FE^2$;	2 ax.
but GF^2+FE^2	=	EG^2,	
$\therefore EG^2$	=	$2\,FE^2$;	
but FE	=	CD,	34. 1.
$\therefore EG^2$	=	$2\,CD^2$.	
Now AE^2	=	$2\,AC^2$,	
$\therefore AE^2+EG^2$	=	$2\,AC^2+2\,CD^2$;	
but AE^2+EG^2	=	AG^2,	
$\therefore AG^2$	=	$2\,AC^2+2\,CD^2$;	
but AG^2	=	AD^2+DG^2,	
$\therefore AD^2+DG^2$	=	$2\,AC^2+2\,CD^2$;	
now DG	=	DB,	
$\therefore AD^2+DB^2$	=	$2\,AC^2+2\,CD^2$.	

Wherefore if a right line, &c. &c. Q. E. D.

PROP. XI.—PROBLEM.

To divide a given right line into two such parts, that the rectangle contained by the whole, and one of the parts, shall be equal to the square of the other part.

Let AB be the given right line; it is required to divide AB into two such parts, that the rectang. contained by the whole and one part shall = square of the other part.



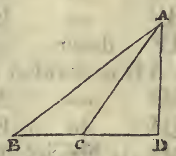
On AB	descr.	sq. AD;	
bis. AC	in	E;	
	join BE;		
prod. CA	to	F;	
and make EF	=	EB;	3. 1.
on AF	descr.	sq. FH;	
	then AB is divided in H		
so that $AB \times BH$	=	AH^2.	
Prod. GH	to	K;	
then ∵ AC	is bis. in	E;	
	and is prod. to F,		
∴ $CF \times FA + AE^2$	=	EF^2;	6. 2.
but EF	=	EB,	constr.
∴ $CF \times FA + AE^2$	=	EB^2;	
but $EA^2 + AB^2$	=	EB^2,	47. 1.
∴ $CF \times FA + AE^2$	=	$AE^2 + AB^2$:	1 ax.
	take away com. AE^2,		
∴ $CF \times FA$	=	AB^2;	3 ax.
but fig. FK	is	$CF \times FA$,	
(for AF	=	FG),	30 def. 1.
also, fig. AD	is	AB^2,	constr.
∴ fig. FK	=	fig. AD;	1 ax.
	take away com. part AK,		
∴ rem. FH	=	rem. HD:	3 ax.
but □ HD	is	$AB \times BH$,	
(for AB	=	BD),	30 def. 1.
also FH	=	AH^2,	constr
∴ $AB \times BH$	=	AH^2.	

Wherefore AB is divided as required. Q. E. F.

PROP. XII.—Theorem.

In obtuse angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle, is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the right line intercepted without the triangle between the perpendicular and the obtuse angle.

Let △ ABC have the obt. ∠ ACB. And from A let fall AD ⊥ BC produced. Then $AB^2 > BC^2 + CA^2$ by $2\ BC \times CD$.



$\because$ BD is div. in C,

$\therefore BD^2 = BC^2 + CD^2 + 2\ BC \times CD$; 4. 2.

add AD^2,

$\therefore BD^2 + AD^2 = BC^2 + CD^2 + AD^2 + 2\ BC \times CD$; 2 ax.

but $AB^2 = BD^2 + AD^2$, 47. 1.

(for ∠ D is rt. ∠); hyp.

Similarly, also $AC^2 = AD^2 + DC^2$,

$\therefore AB^2 = BC^2 + CA^2 + 2\ BC \times CD$;

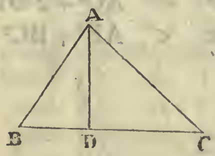
i. e. $AB^2 > BC^2 + CA^2$ by $2\ BC \times CD$.

Wherefore in obtuse angled, &c. &c. &c. Q. E. D.

PROP. XIII.—THEOREM.

In every triangle, the square of the side subtending either of the acute triangles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the right line intercepted between the perpendiculars let fall upon it from the opposite angle, and the acute angle.

Let $\triangle$ ABC have the acute $\angle$ ABC, and let fall from opp. $\angle$ AD $\perp$ BC one of the sides cont. $\angle$ B. Then $AC^2 < CB^2 + BA^2$ by $2\ CB \times BD$.



FIRST—let AD fall within $\triangle$ ABC.

and $\because$ BC is divid. in D,

$$\therefore BC^2 + BD^2 = 2\,BC \times BD + DC^2; \qquad \text{7. 2.}$$

add AD^2,

$$\therefore BC^2 + BD^2 + AD^2 = 2BC \times BD + AD^2 + DC^2; \qquad \text{2 ax.}$$

$$\text{but } AB^2 = AD^2 + DB^2, \qquad \text{47. 1.}$$

(for $\angle$ ADB is a rt. $\angle$), hyp.

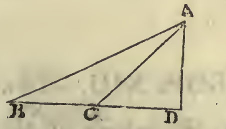
$$\text{Similarly, also } AC^2 = AD^2 + DC^2, \qquad \text{47. 1.}$$

$$\therefore AB^2 + BC^2 = 2\,BC \times BD + AC^2; \qquad \text{1 ax.}$$

$$\text{i.e. } AC^2 \text{ alone} < CB^2 + BA^2 \text{ by } 2\,BC \times BD.$$

SECONDLY,

PROP. XIII. CONTINUED.



SECONDLY—let AD fall without △ ABC;

then, ∵ ∠ D is a rt. ∠, — hyp.

* ∴ ∠ ACB > rt. ∠; — 16. 1.

and ∴ $AB^2 = AC^2 + CB^2 + 2\ BC \times CD$; — 12. 2.

add BC^2,

∴ $AB^2 + BC^2 = AC^2 + 2CB^2 + 2BC \times CD$; — 2 ax.

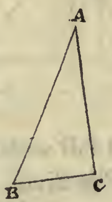
but ∵ BD is ÷ in C,

∴ $DB \times BC = BC \times CD + BC^2$; — 3. 2.

and ∴ $2\ DB \times BC = 2\ BC \times CD + 2\ BC^2$, — 2 ax.

∴ $AB^2 + BC^2 = AC^2 + 2\ DB \times BC$,

∴ AC^2 alone $< AB^2 + BC^2$ by $2\ DB \times BC$.



LASTLY—let the side AC ⊥ BC;

then BC is the rt. line between the ⊥ and acute ∠ B;

and it is manifest that $AB^2 + BC^2 = AC^2 + 2BC^2$. 47.1. & 2 ax.

Wherefore in every triangle, &c. &c. Q. E. D.

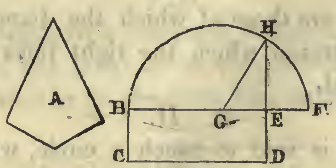
---

* For ∠ ACB is the exterior ∠ of the △ ACD; and ∴ greater than the interior ∠ ADC.

PROP. XIV.—Problem.

To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal fig. It is required to descr. a sq. = fig. A.



Descr. rt. ∠d □ BD = fig. A. 45. 1.
Then if BE = ED,
∴ BD is a sq.; 30 def. 1.
and that which was required is done.
But if BE ≠ ED;
prod. BE to F;
make EF = ED;
bis. BF in G; 10. 1.
with cent. G, and dist. GB or GF descr. $\frac{1}{2}$ ⊙ BHF;
prod. DE to H.
Then EH^2 = rtlin. fig. A.
Join GH;
and ∵ BF is bis. in G,
and divided into two unequal parts in E,
∴ $BE \times EF + EG^2 = GF^2$; 5. 2.
but GF = GH, 15 def. 1.
∴ $BE \times EF + EG^2 = GH^2$; 1 ax.
but $HE^2 + EG^2 = GH^2$, 47. 1.
∴ $BE \times EF + EG^2 = HE^2 + EG^2$;
take away com. EG^2,
∴ rem. $BE \times EF = EH^2$; 3 ax.
but, $BE \times EF$ = □ BD,
(for EF = ED), constr.
∴ □ BD = EH^2;
but □ BD = rtlin. fig. A, constr.
∴ rectil. fig. A = EH^2.

Wherefore the sq. described on EH = given rectil. fig. A.
Q. E. F.

BOOK III.

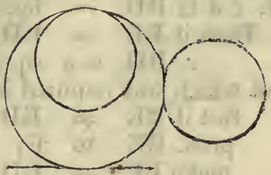
DEFINITIONS.

I.

Equal circles are those of which the diameters are equal, or from the centres of which the right lines to the circumference are equal.

II.

A right line is said to touch a circle, when it meets the circle, and being produced does not cut it.

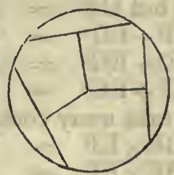


III.

Circles are said to touch each other, which meet, but do not cut each other.

IV.

Right lines are said to be equally distant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



V.

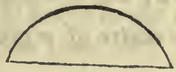
And the right line on which the greater perpendicular falls, is said to be farther from the centre.

A

An arc is any part of the circumference of a circle.

VI.

A segment of a circle is a figure contained by a right line, and the circumference which it cuts off.



VII.

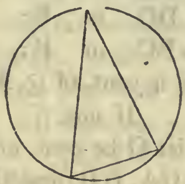
The angle of a segment is that which is contained by the right line and the circumference.

VIII.

An angle in a segment is the angle contained by two right lines drawn from any point in the circumference of the segment to the extremities of the right line which is the base of the segment.

IX.

An angle is said to stand on the circumference intercepted between the right lines that contain the angle.

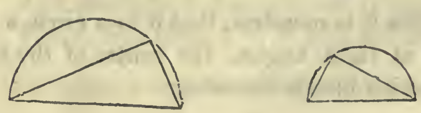


X.

A sector of a circle is the figure contained by two right lines drawn from the centre, and the circumference between them.

XI.

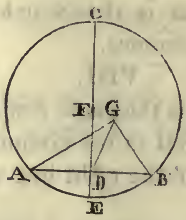
Similar segments of circles are those in which the angles are equal, or which contain equal angles.



PROP. I.—Problem.

To find the centre of a given circle.

Let ABC be the given ⊙ ; it is required to find its centre.



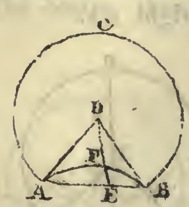
Draw within ⊙ ABC any right line AB ;
bis. AB in D ; 10. 1.
from D, draw DC at rt. ∠ s to AB ; 11. 1.
prod. DC to E ;
bis. EC in F ;
Then F is cent. of ⊙ ABC.
If not,
if possible, let G be cent. of ⊙ ABC ;
Join GA, GD, and GB ;
and ∵ DA = DB, constr.
and DG com. to △ s ADG, BDG,
and that base BG = base AG, 15 def. 1.
∴ ∠ ADG = ∠ BDG ; 8. 1.
and ∴ ∠ BDG is a rt. ∠ ; 10 def. 1.
but also ∠ FDB is a rt. ∠ ; constr.
∴ ∠ FDB = ∠ BDG, 1 ax.
i. e. greater = less,
which is impossible.
∴ G is not cent. ⊙ ABC.
Similarly none but F is cent. of ⊙ ABC.
Therefore F is cent. of ⊙ ABC. Q. E. F.

Cor. From this it is manifest, that if in a circle a right line bisect another at right angles, the centre of the circle is in the right line which bisects the other.

PROP. II.—Theorem.

If any two points be taken in the circumference of a circle, the right line which joins them shall fall within the circle.

Let ABC be a ⊙, and let any points A and B be taken in ⊙. The right line drawn from A to B shall fall within the ⊙.



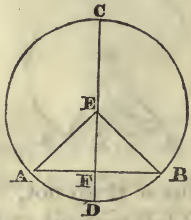
For if it do not,
if possible, let AB fall without ⊙ ABC as AEB;
find D cent. ⊙ ABC; 1. 3.
and join DA, DB;
in $\widehat{AB}$ take any pt. F;
join DF;
prod. DF to E.
Then ∵ DA = DB, 15 def. 1.
∴ ∠ DAB = ∠ DBA: 5. 1.
and ∵ ∠ DEB is the ex. ∠ of △ DAE,
∴ ∠ DEB > ∠ DAE; 16. 1.
but ∠ DBE = ∠ DAE,
∴ ∠ DEB > ∠ DBE; 1 ax.
and ∴ DB > DE; 19. 1.
but DB = DF, 15 def. 1.
∴ DF > DE:
i. e. less > greater.
which is impossible.
∴ The rt. line from A to B does not fall without the ⊙.
And similarly it does not fall upon the ⊙.
∴ The rt. line from A to B falls within ⊙ ABC.

Wherefore if any two points, &c. &c. Q. E. D.

PROP. III.—Theorem.

If a right line drawn through the centre of a circle, bisect a right line in it which does not pass through the centre, it shall cut it at right angles; and if it cut it at right angles, it shall bisect it.

First.—Let CD passing through cent. of ⊙ ABC bis. any right line AB, which does not pass through the centre, in F; it shall cut AB at right ∠s.



Take E cent. of ⊙ ABC; — 1. 3.
Join EA, EB,
Then ∵ AF = FB, — hyp.
and FE com. to △s AFE, BFE,
and that base EA = base EB, — 15 def. 1.
∴ ∠ AFE = BFE; — 8. 1.
and ∴ each of ∠s AFE, BFE is a rt. ∠; — 10 def. 1.
∴ CD cuts AB at rt. ∠s.

Secondly.—Let CD cut AB at right ∠s; CD shall also bis. AB.

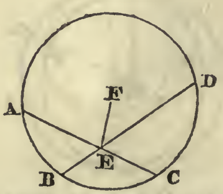
The same constr. being made.
∵ EA = EB, — 15 def. 1.
∴ ∠ EAF = ∠ EBF. — 5. 1.
And rt. ∠ AFE = rt. ∠ BFE, — 11. ax.
∴ in the △s EAF, EBF,
∠ EAF = ∠ EBF,
and ∠ AFE = ∠ BFE,
also opp. side EF is com. to the △s,
∴ AF = FB. — 26. 1.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. IV.—Theorem.

If, in a circle, two right lines, not passing through the centre, cut each other, they do not bisect each other.

Let ABCD be a circle, and AC, BD two right lines in it not passing through the centre, they shall not bisect each other.



For if possible let AE = EC,
and BE = ED;

If one of the lines pass through cent. it is evident that it cannot be bis. by the other which does not pass through cent. But if neither of them pass through cent.

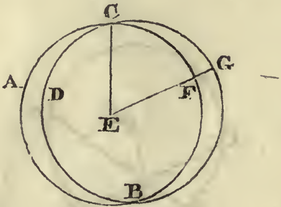
take F cent. ⊙ — 1. 3.
Join EF,
and ∵ EF thro. cent bis. AC not thro. cent. — hyp.
∴ EF is at rt. ∠ s to AC; — 3. 3.
∴ ∠ FEA is a rt. ∠.
Similarly ∵ FE thro. cent. bis. BD not thro. cent. — hyp.
∴ FE is at rt. ∠ s to BD; — 3. 3.
∴ ∠ FEB is a rt. ∠;
but ∠ FEA is a rt. ∠,
∴ ∠ FEA = ∠ FEB; — 1 ax.
i. e. less = greater,
which is impossible:
∴ AC, BD do not bis. each other.

Wherefore if in a circle, &c. &c. Q. E. D.

PROP. V.—THEOREM.

If two circles cut each other, they shall not have the same centre.

Let ⊙s ABC, CDG cut each other in pts. C and B; they shall not have the same centre.



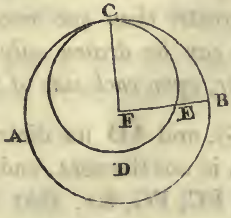
For, if possible, let E be com. cent. to both.
Join EC;
and draw any rt. line EFG meeting ⊙s in F and G;
and ∵ E is cent. of ⊙ ABC,
∴ EC = EF. 15 def. 1.
Again ∵ E is cent. of ⊙ CDG,
∴ EC = EG; 15 def. 1.
but EC = EF,
∴ EF = EG; 1 ax.
i. e. less = greater,
which is impossible.
∴ E is not a com. cent. to ⊙s ABC, CDG.

Wherefore if two circles cut each other, &c. &c. Q. E. D.

PROP. VI.—Theorem.

If two circles touch each other internally, they shall not have the same centre.

Let two ⊙s ABC, CDE touch each other in pt. C, they shall not have the same centre.



If possible let F be a com. cent.

Join FC;

and draw any rt. line FEB meeting ⊙s in E and B.

Then, ∵ F is cent. of ⊙ ABC,

∴ FC = FB. 15 def. 1.

Again, ∵ F is cent. of ⊙ CDE,

∴ FC = FE; 15 def. 1.

but FC = FB,

∴ FE = FB; 1 ax.

i. e. less = greater;

which is impossible.

∴ F is not cent. of ⊙s ABC, CDE.

Therefore if two circles touch each other internally, &c. &c. Q. E. D.

PROP. VII.—Theorem.

If any point be taken in the diameter of a circle which is not the centre, of all the right lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least, and, of any others, that which is nearer to the line which passes through the centre is always greater than one more remote; and from the same point there can be drawn only two right lines that are equal to each other, upon each side of the shortest line.

Let ABCD be a ⊙, and AD its diam. in which let any pt. F be taken which is not the cent. and let cent. be E. Of all the rt. lines FB, FC, FG, &c. that can be drawn from F to ○, FA shall be greatest, and FD shall be the least; and of the others FB shall be > FC, and FC > FG, &c.

Join BE, CE, GE.			
Then, ∵ in the △ BEF,			
BE + EF	>	BF,	20. 1.
and that AE	=	BE,	15 def. 1.
∴ AE + EF, i. e. AF	>	BF.	
And ∵ BE	=	CE,	
and FE is com. to △s, BEF, CEF,			
∴ BE, EF	=	CE, EF, ea. to ea.	
also ∠ BEF	>	∠ CEF,	9 ax.
∴ base BF	>	base CF.	24. 1.
Similarly CF	>	GF.	

And

PROP. VII.—CONTINUED.

Again, ∵ GF + FE > EG, 20. 1.
and EG = ED,
∴ GF + FE > ED;
take away com. FE,
∴ rem. GF > FD; 5 ax.

∴ AF is the greatest } of all rt. lines drawn from F to ○.
and FD is the least }

Also BF > CF,
and FC > FG.

Also there can be drawn only two equal rt. lines from pt. F to ○, one on each side of the shortest line FD.

At E in EF make ∠ FEH = ∠ FEG; 23. 1.
Join FH.
Then, ∵ GE = EH, 15 def. 1.
and that EF is com. to △s GEF, HEF,
and that ∠ GEF = ∠ HEF, constr.
∴ base FG = base FH. 4. 1.

And besides FH no other rt. line can be drawn from F to ○, = FG;
for, if there can, let it be FK:
and ∵ FK = FG,
and FG = FH,
∴ FK = FH; 1 ax.

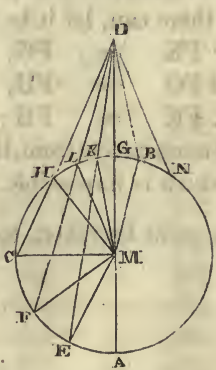
i. e. a line near to, = one more remote from, that passing thro. cent.
which is impossible.

Therefore if any point be taken, &c. &c. Q. E. D.

PROP. VIII.—Theorem.

If any point be taken without a circle, and right lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall on the concave circumference, the greatest is that which passes through the centre, and of the rest, that which is nearer to the one passing through the centre, is always greater than one more remote; but, of those which fall on the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote: and only two equal right lines can be drawn from the same point to the circumference, one on each side of the least line.

Let ABC be a ⊙ and D any pt. without it, from which let DA, DE, DF, DC, be drawn to ○, whereof, DA passes through the cent. Of those which fall on the concave ○, the greatest shall be DA. And the one nearer to DA shall be > one more remote, viz. DE > DF > and DF > DC. But of those which fall on the convex ○ HLKG the least shall be DG, between pt. D and diam. AG; and the nearer to it shall be < one more remote; viz. DK < DL and DL < DH.



Take M cent. of ⊙ ABC;
join ME, MF, MC, MH, ML, MK;
and ∵ MA = EM, 15 def. 1.
add MD,
∴ AD = ME + MD; 2 ax.
but EM + MD > ED, 20. 1.
∴ AD > ED.

Again,

PROP. VIII.—CONTINUED.

Again, ∵ ME = MF, 15 def. 1.
and MD is com. to △s EMD, FMD,
and that ∠ EMD > ∠ FMD, 9 ax.
∴ base DE > base DF. 24. 1.
Similarly DF > DC,
∴ of all the rt. lines drawn from D to concave ○,
AD > any of them;
and also DE > DF;
and DF > DC.
Again, ∵ MK + KD > MD, 20. 1.
and MK = MG, 15 def. 1.
∴ rem. KD > rem. GD; 5 ax.
i. e. GD < KD.
And ∵ MLD is a △,
and that, from M, D extrems. of its side MD are drawn MK, KD to pt. K within it,
∴ MK + KD < ML + LD; 21. 1.
but MK = ML, 15 def. 1.
∴ rem. DK < rem. DL. 5 ax.
Similarly DL < DH;
∴ of all the rt. lines drawn from D to convex ○,
DG < any other;
also DK < DL;
and DL < DH.

Also there can be drawn only two equal rt. lines from D to ○, i. e. one on each side of least line.

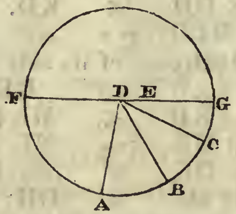
At M in MD make ∠ DMB = ∠ DMK; 23. 1.
and join DB.
And ∵ MK = MB, 15 def. 1.
and MD com. to △s KMD, BMD,
and that ∠ KMD = BMD, constr.
∴ base DK = base DB;
and besides DB, none other can be drawn from D to ○, = DK.
For, if there can, let it be DN;
and ∵ DK = DN,
and that also DK = DB,
∴ DB = DN;
i. e. a line nearer to the least = one more remote,
which is impossible.

Wherefore if any point, &c. &c. Q. E. D.

PROP. IX.—THEOREM.

If a point be taken within a circle, from which there fall more than two equal right lines to the circumference, that point is the centre of the circle.

Let the pt. D be taken in ⊙ ABC, from which to the ○ there fall more than two equal rt. lines, viz. DA, DB, DC; the point D shall be cent. of ⊙.

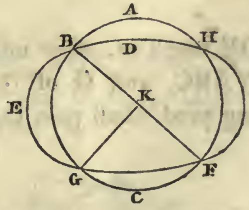


For, if not, let E be cent. of ⊙ ABC;
Join DE;
prod. DE both ways to ○ in F, G;
then FG is diam.
And ∵ a pt. D, not the cent. is taken in diam. FG,
∴ DG is > any other rt. line drawn from D to ○; 7. 3.
also, DC > DB;
and DB > DA;
but DA, DB, DC = each other; hyp.
which is impossible.
∴ E is not cent. of ⊙ ABC.
Similarly, none but D is cent. ⊙ ABC;
∴ D is cent. ⊙ ABC.

Wherefore if a point be taken, &c. &c. Q. E. D.

PROP. X.—Theorem.

One circumference of a circle cannot cut another in more than two points.



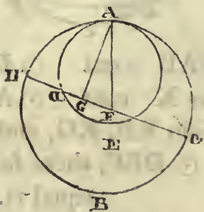
If possible, let ○ FAB cut ○ DEF in pts. B, G, F.
Take K cent. ⊙ ABC; 1. 3.
and join KB, KG, and KF.
And ∵ from pt. K, in ⊙ DEF, there fall to ○ more than two equal rt. lines KB, KG, KF,
∴ K is cent. ⊙ DEF; 9. 3.
but K is cent. ⊙ ABC; constr.
∴ same point is cent. of 2 ⊙s which cut each other;
which is impossible. 5. 3.

Therefore one circumference, &c. &c. Q. E. D.

PROP. XI.—Theorem.

If two circles touch each other internally, the right line which joins their centres, being produced, shall pass through the point of contact.

Let ⊙s, ABC, ADE touch ea. other intern. in pt. A. And let F be cent. of ⊙ ABC, and G of ⊙ ADE; the rt. line joining F and G, being prod. shall pass thro. pt. of contact A.



If not, let it fall otherwise, if possible, as GD.

Join AF, AG;

and ∵, in the △ AGF,

FG + GA	>	FA,	20. 1.
and FA	=	FH,	15 def. 1.
∴ FG + GA	>	FH;	
take away com. FG,			
∴ rem. GA	>	GH;	
but GA	=	GD,	15 def. 1.
(for G is cent. of ⊙ ADE),			
∴ GD	>	GH;	
i. e. less	>	greater;	

which is impossible.

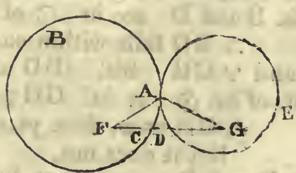
∴ The rt. line joining cents. F and G, being prod. must fall on A; i. e. it must pass thro. A.

Wherefore if two circles, &c. &c. Q. E. D.

PROP. XII.—Theorem.

If two circles cut each other externally, the right line which joins their centres, shall pass through the point of contact.

Let ⊙ ABC touch ⊙ ADE extern. in pt. A. And let F be cent. of ⊙ ABC, and G of ⊙ ADE; the rt. line joining F and G shall pass thro. A.



For, if not, let it fall otherwise, if possible, as FCDG.

Join FA, AG.

And ∵ F is cent. ⊙ ABC,

∴ FA = FC : 15 def. 1.

also, ∵ G is cent. ⊙ ADE,

∴ GA = GD ; 15 def. 1.

∴ FA + AG = FC + DG ; 2 ax.

∴ whl. FG > FA + AG ;

but also FG < FA + AG, 20. 1.

which is impossible.

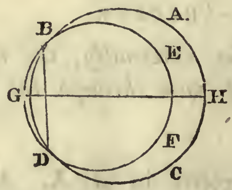
∴ The rt. line joining cents. F and G must fall on pt. of contact A.

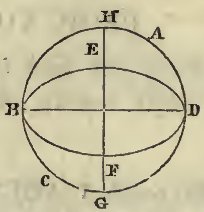
Wherefore if two circles, &c. &c. Q. E. D.

PROP. XIII.—Theorem.

One circle cannot touch another in more points than one, whether it touches it internally or externally.

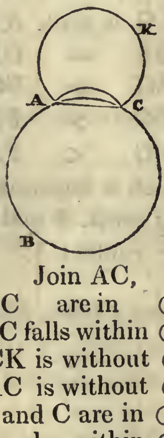
First.—If possible, let ⊙ EBF touch ⊙ ABC internally in pts. B and D.





Join BD;
draw GH, bisecting BD at rt. ∠s. 10. 11. 1.
Then, ∵ pts. B and D are in ○ of ea. ⊙,
∴ BD falls within ea. ⊙; 2. 3.
and ∵ GH bis. BD at rt. ∠s,
∴ cent. of ea. ⊙ is in GH; cor. 1.3.
∴ GH pass. thro. pt. of contact; 11. 3.
but it does not,
for pts. B and D are not in rt. line GH;
which is absurd,
and ∴ one circle cannot touch another *internally* in more than one point.

Secondly.—If possible, let ⊙ ACK touch ⊙ ABC externally in pts. A and C.



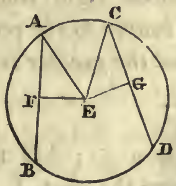
Join AC,
And ∵ pts. A and C are in ○ of ⊙ ACK,
∴ AC falls within ⊙ ACK; 2. 3.
but ⊙ ACK is without ⊙ ABC,
∴ AC is without ⊙ ABC;
but ∵ pts. A and C are in ○ of ⊙ ABC,
∴ AC is also within ⊙ ABC.
which is absurd.
and ∴ one circle cannot touch another *externally* in more points than one.

Wherefore one circle, &c. &c. Q. E. D.

PROP. XIV.—Theorem.

Equal right lines in a circle are equally distant from the centre: and those which are equally distant from the centre, are equal to each other.

First—In ⊙ ABDC let AB = CD; they shall be equally dist. from cent.



Take E cent. ⊙ ABDC; 1. 3.
from E draw EF ⊥ AB; and EG ⊥ CD; 12. 1.
then, ∵ EF thro. cent. is at rt. ∠s to AB not thro. cent.
$\therefore AF = FB$; 3. 3.
and $\therefore AB = 2\,AF$;
similarly, $CD = 2\,CG$;
but $AB = CD$, hyp.
$\therefore AF = CG$;
and $\because AE = EC$, 15 def. 1.
$\therefore AE^2 = EC^2$;
but $AF^2 + FE^2 = AE^2$, 47. 1.
(for ∠ AFE is a rt. ∠); constr.
similarly, $EG^2 + GC^2 = EC^2$,
$\therefore AF^2 + FE^2 = EG^2 + GC^2$. 1 ax.
Now $AF^2 = CG^2$,
$\therefore$ rem. $FE^2 =$ rem. EG^2, 3 ax.
and $\therefore FE = EG$;
and FE, EG are drawn from cent. E at rt. ∠s to AB and CD, [constr.
∴ AB and CD are equally dist. from cent. 4 def. 3.

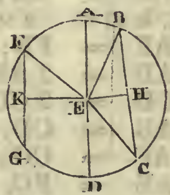
Secondly—Let AB, CD be equally dist. from the cent. i. e. $FE = EG$: then $AB = CD$.
$\because AF^2 + FE^2 = EG^2 + GC^2$, demon.
of which $FE^2 = EG^2$,
(for $FE = EG$), hyp.
$\therefore$ rem. $AF^2 =$ rem. GC^2; 3 ax.
$\therefore AF = CG$;
but $AB = 2\,AF$,
and $CD = 2\,CG$,
$\therefore AB = CD$. 6 ax.

Wherefore equal right lines, &c. &c. Q. E. D.

PROP. XV.—Theorem.

The diameter is the greatest right line in a circle; and of any others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

First—Let ABCD be a $\odot$; AD the diam. and E the cent. and let BC be nearer the cent. E than FG; then shall AD > BC, and BC > FG.



From E draw EH, EK $\perp$ BC and FG; 12. 1.

Join EB, EC, EF.

and $\because$ AE = EB,
and ED = EC; } 15 def. 1.

$\therefore$ AD = BE + EC;

but BE + EC > BC, 20.1.

$\therefore$ AD > BC:

and $\because$ BC is nearer cent. than FG, hyp.

$\therefore$ EH < EK;

but BC = 2 BH,*
and FG = 2 FK; } 14. 3.

---

* For, EH thro. cent. E is at rt. $\angle$s to BC not thro. cent.

$\therefore$ BH = HC; 3. 3.

and $\therefore$ BC = 2 BH:

Similarly, FG = 2 FK.

PROP. XV. CONTINUED.

and $EH^2 + HB^2$	$=$	$EK^2 + KF,^2$*	
of which EH^2	$<$	EK^2,	
(for EH	$<$	EK,)	hyp.
$\therefore HB^2$	$>$	FK^2;	
and $\therefore$ HB	$>$	FK;	
$\therefore$ whl. BC	$>$	whl. FG.	

SECONDLY—Let BC > FG; then shall BC be nearer to the cent. than FG. i. e. EH < EK.

for $\because$ BC	$>$	FG,	hyp.
$\therefore$ BH	$>$	FK;	
and $BH^2 + HE^2$	$=$	$FK^2 + KE^2$,	
of which BH^2	$>$	FK^2,	
$\therefore EH^2$	$<$	EK^2;	
and $\therefore$ EH	$<$	EK;	
and $\therefore$ BC is nearer cent. than FG.			5 def. 3.

Wherefore the diameter, &c. &c. Q. E. D.

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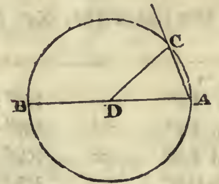
* For, EF^2	$=$	EB^2;	15 def. 1; and 2 ax.
but EF^2	$=$	$FK^2 + KE^2$,	47. 1.
and EB^2	$=$	$EH^2 + HB^2$,	47. 1.
and $\therefore EH^2 + HB^2$	$=$	$EK^2 + KF^2$.	1 ax.

PROP. XVI.—Theorem.

The right line which is drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no right line can be drawn from the extremity, between that right line and the circumference which does not cut the circle, or, which is the same thing, no right line can make so great an acute angle with the diameter at its extremity, or so small an angle with the right line which is at right angles to it, as not to cut the circle.

First.—Let ABC be the ⊙; AB diam. and D cent. The rt. line drawn from the extremity A at rt. ∠s to AB shall fall without ⊙ ABC.



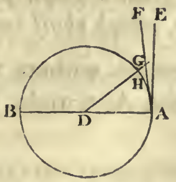
For, if not, let it, if possible, fall within ⊙ as AC.

Draw DC to pt. C where AC meets ○;
Then ∵ DA = DC, 15 def. 1.
∴ ∠ DAC = ∠ ACD; 5. 1.
but ∠ DAC is a rt. ∠, hyp.
∴ ∠ ACD is a rt ∠;
∴ in △ ACD; 2 ∠ s, i.e. ACD + DAC = 2 rt. ∠ s;
which is impossible.
∴ AC does not fall within ⊙;
Similarly AC does not fall on the ○;
∴ AC falls without the ⊙ ABC as AE.

Secondly.

PROP. XVI. CONTINUED.

SECONDLY—Between AE and ○ no rt. line can be drawn from A which does not cut ⊙.



For, if possible, let FA be between them.			
From D draw DG	⊥	FA;	12. 1.
and let DG	meet	○ in H:	
and ∵ ∠ AGD	is a	rt. ∠,	constr.
and ∠ DAG	<	rt. ∠,	9 ax.
∴ DA	>	DG;	19. 1.
but DA	=	DH,	
∴ DH	>	DG;	
i. e. less	>	greater.	
which is impossible.			

Therefore no right line can be drawn from A between AE and ○ which does not cut the ⊙; or, which amounts to the same thing, however great an acute angle a right line makes with the diameter at A, or however small with AE, the ○ shall pass between that right line and the perpendicular AE. "And this is all that is to be understood, when in the Greek "text, and in translations from it, the angle of the semicircle "is said to be greater than any acute rectilineal angle, and "the remaining angle less than the rectilineal angle."

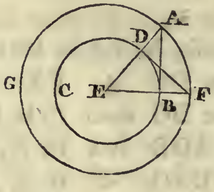
Cor. From this it is manifest that the right line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, because if it did meet the circle in two, it would be within it.* "Also it is evident that there can be but "one right line which touches the circle in the same point."

*2. 3.

PROP. XVII.—Problem.

To draw a right line from a given point, either without or within the circumference, which shall touch a given circle.

First—Let A be given pt. without the given circle BCD; it is required to draw from A, a right line which shall touch ⊙ BCD.



Find E cent. ⊙ BCD; 1. 3.
join AE;
with cent. E, and dist. EA descr. ⊙ AFG;
from D draw DF at rt. ∠ s to EA; 11. 1.
join EF, AB;
then shall AB touch ⊙ BCD.
For ∵ E is cent. ⊙s BCD, AFG,
∴ EB = ED, } 15 def. 1.
and EF = EA, }
∴ AE, EB = FE, ED ea. to ea.
and they contain an ∠ E com. to △ s AEB, FED,
∴ base DF = base AB, }
and △ FED = △ AEB, } 4. 1.
and ∠ EDF = ∠ EBA; }
but ∠ EDF is a rt. ∠, constr.
∴ ∠ EBA is a rt. ∠.
∴ AB, drawn from extrem. B, is rt. ∠ s to diam. EB;
∴ AB touches ⊙ BCD; 16. 3. cor.
and it is drawn from the given point A.

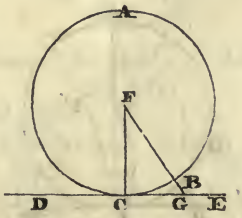
Secondly—Let the given pt. be within the ○ of the ⊙ as D.

Draw DE to cent. E;
and DF at rt. ∠ s to DE;
then DF touches ⊙. cor. 16. 3.

PROP. XVIII.—Theorem.

If a right line touch a circle, the right line drawn from the centre to the point of contact, shall be perpendicular to the line which touches the circle.

Let DE touch ⊙ ABC in C; and let FC be drawn from cent. F to C, the pt. of contact; then shall FC ⊥ DE.



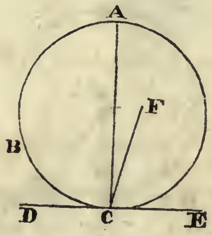
For, if FC is not ⊥ DE;			
draw FG	⊥	DE.	12. 1.
then, ∵ FGC	is a	rt. ∠,	
∴ GCF	<	rt. ∠;	17. 1.
∴ ∠ FGC	>	∠ GCF;	
and ∴ also FC	>	FG;	19. 1.
but FC	=	FB,	15 def. 1.
∴ FB	>	FG;	
i. e. less	>	greater.	
which is impossible.			
∴ FG is not ⊥ DE;			
Similarly, none but FC	⊥	DE;	
∴ FC	⊥	DE.	

Therefore if a right line, &c. &c. Q. E. D.

PROP. XIX.—Theorem.

If a right line touches a circle, and from the point of contact a right line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let DE touch ⊙ ABC in C, and let AC be drawn from C at rt. ∠ s to DE; the centre of ⊙ shall be in AC.



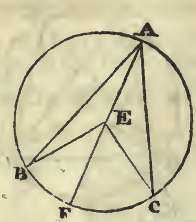
For, if not, if possible, let F be cent. ⊙ ABC.
Join CF;
and ∵ DE touches ⊙ ABC,
and FC is drawn from cent. to pt. of contact,
∴ FC ⊥ DE; 18. 3.
and ∴ ∠ FCE is a rt. ∠;
but ∠ ACE is a rt. ∠,
∴ ∠ FCE = ∠ ACE;
i. e. less = greater,
which is impossible.
∴ F not cent. ⊙ ABC,
Similarly, none other pt. without AC is cent. ⊙ ABC;
i. e. the cent. is in AC.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. XX.—Theorem.

The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same part of the circumference.

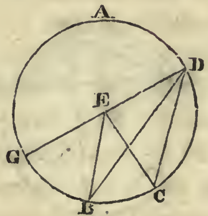
In $\odot$ ABC let $\angle$ BEC be at cent. E. and $\angle$ BAC at $\bigcirc$, having same part of $\bigcirc$, BC, for their base. Then shall $\angle$ BEC $= 2 \angle$ BAC.



First—Let cent. E be within $\angle$ BAC.

Join AE;

prod. AE	to	F:	
and $\because$ EA	$=$	EB,	15 def. 1.
$\therefore \angle$ EAB	$=$	$\angle$ EBA;	5. 1.
$\therefore \angle$s EAB + EBA	$=$	$2 \angle$ EAB;	
but $\angle$ BEF	$=$	$\angle$s EAB + EBA,	32. 1.
$\therefore \angle$ BEF	$=$	$2 \angle$ EAB;	1 ax.
Similarly, $\angle$ FEC	$=$	$2 \angle$ EAC;	
$\therefore$ whl. $\angle$ BEC	$=$	2 whl. $\angle$ BAC.	



Secondly—Let cent. E be without $\angle$ BDC.

Join DE;

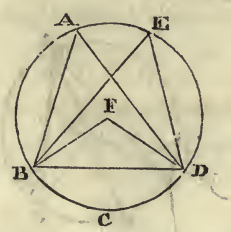
prod. DE	to	G:	
and $\because$ EC	$=$	ED,	15 def. 1.
$\therefore \angle$ EDC	$=$	$\angle$ ECD;	5. 1.
and $\therefore \angle$s EDC + ECD	$=$	$2 \angle$ EDC;	
but $\angle$ GEC	$=$	$\angle$s, EDC + ECD,	32. 1.
$\therefore \angle$ GEC	$=$	$2 \angle$ EDC:	
Similarly, part $\angle$ GEB	$=$	2 part $\angle$ GDB;	
$\therefore$ rem. $\angle$ BEC	$=$	2 rem. $\angle$ BDC.	

Therefore the angle, &c. &c. Q. E. D.

PROP. XXI.—Theorem.

The angles in the same segment of a circle are equal to each other.

Let $\angle$s BAD, BED be in same seg. BAED. Then shall $\angle$ BAD $=$ $\angle$ BED.



Take F cent. $\odot$ ABCD.

First—Let the seg. be $>$ $\frac{1}{2}\odot$.

Join FB, FD:

and $\because$ $\angle$ BFD is at cent. F,

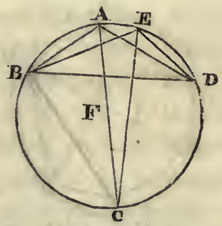
and that $\angle$ BAD is at $\bigcirc$,

and that both have same base BD,

$\therefore$ $\angle$ BFD $=$ $2\angle$ BAD: 20.3.

Similarly, $\angle$ BFD $=$ $2\angle$ BED;

$\therefore$ $\angle$ BAD $=$ $\angle$ BED.



Secondly—Let the seg. be $<$ $\frac{1}{2}\odot$.

Draw AC through cent. F;

join CE;

$\therefore$ seg. BADC $>$ $\frac{1}{2}\odot$,

and the $\angle$s in it are equal,

i. e. $\angle$ BAC $=$ $\angle$ BEC: 1st case.

Similarly, $\angle$ CAD $=$ $\angle$ CED;

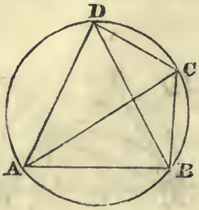
$\therefore$ whl. $\angle$ BAD $=$ whl. $\angle$ BED.

Wherefore the angles, &c. &c. Q. E. D.

PROP. XXII.—Theorem.

The opposite angles of any quadrilateral figure described in a circle, are together equal to two right angles.

Let the quadrilat. fig. ABCD be inscribed in ⊙ ABCD; any two of its opposite $\angle$s together = 2 rt. $\angle$s.



Join AC, BD.

Now ∵ $\angle$s BAC, BDC are in same seg. BADC,

∴ $\angle$ BAC = $\angle$ BDC: 21. 3.

Similarly, $\angle$ ADB = $\angle$ ACB;

∴ whl. $\angle$ ADC = $\angle$s BAC + ACB;

add $\angle$ CBA,

∴ $\angle$s ADC + CBA = $\angle$s CBA + BAC + ACB;

but $\angle$s CBA + BAC + ACB = 2 rt. $\angle$s, 32. 1.

∴ $\angle$s ADC + CBA = 2 rt. $\angle$s;

Similarly, $\angle$s BAD + DCB = 2 rt. $\angle$s;

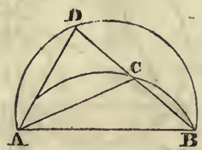
and these are the opposite $\angle$s of the quadrilat. fig, ABCD.

Therefore the opposite angles, &c. &c. Q. E. D.

PROP. XXIII.—Theorem.

Upon the same right line, and upon the same side of it, there cannot be two similar segments of circles, which do not coincide with each other.

If it be possible, let the similar segments ACB, ADB be on the same rt. line AB on the same side of it, and not coincide with each other.



Then ∵ ⊙ ACB cuts ⊙ ADB in the pts. A and B,
it cannot cut it in any other pt. 10. 3.
∴ one segment must fall within the other.
Let seg. ACB fall within seg. ADB.
Draw rt. line BCD, cutting ⊙s in C, D,
join CA, DA;
and ∵ seg. ACB ∼ *seg. ADB,
∴ ∠ ACB = ∠ ADB; 11 def. 3.
i. e. the ex. ∠ = int. ∠.
which is impossible. 16. 1.

Therefore there cannot be on the same rt. line, &c. &c.
Q. E. D.

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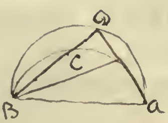
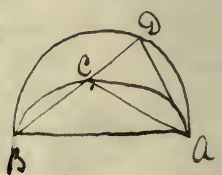
* In writing out the propositions in the Senate House, Cambridge, it will be advisable not to make use of this symbol, but merely to write the word short, thus, *is simil:*





PROP. XXIV.—Theorem.

Similar segments of circles upon equal right lines are equal to each other.

Let the seg. AEB be similar to the seg. CFD, and let them be on equal rt. lines AB, CD : then shall seg. AEB = seg. CFD.

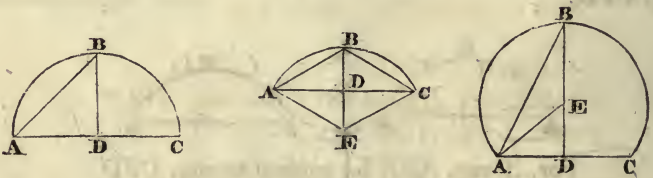
For, if seg. AEB be applied to seg. CFD,
so that pt. A be in pt. C,
and rt. line AB on CD;
then, ∵ AB = CD, hyp.
∴ shall B coinc. with D;
∴ AB coinciding with CD,
the seg. AEB must coin. with seg. CFD; 23.3.
and ∴ seg. AEB = seg. CFD.

Wherefore similar segments, &c. &c. Q. E. D.

PROP. XXV. PROBLEM.

A segment of a circle being given, to describe the circle of which it is the segment.

Let ABC be the given segment; it is required to describe the ⊙ of which it is the segment.



Bisect AC in D;
from D draw BD at rt. ∠s to AC;
Join AB;
FIRST.—Let ∠ ABD = ∠ BAD,
then BD = DA. 6. 1.
∵ DA, DB, DC = ea. other,
∴ D is cent. ⊙; 9. 3.
∴ with cent. D and dist. DA, DB, or DC descr. a ⊙;
and this ⊙ shall pass thro. extrems. of the other two rt. lines;
and the ⊙, of which ABC is a seg. shall be described.

SECONDLY.—Let ∠ ABD ≠ ∠ BAD.
At A in AB, make ∠ BAE = ∠ ABD; 23. 1
prod. BD to E;
and join EC;
and ∵ ∠ ABE = ∠ BAE,
∴ AE = EB; 6. 1.
and ∵ AD = DC, constr.
and DE is com. to △s ADE, CDE,
and that ∠ ADE = ∠ CDE,
∴ base AE = base EC; 4. 1.

but

PROP. XXV.—CONTINUED.

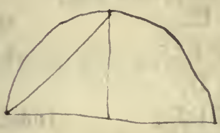
but AE = EB,
∴ AE, EB, EC = ea. other;
∴ is E cent. ⊙.
∴ with cent. E and dist. AE, EB, or EC descr. a ⊙;
and this ⊙ shall pass thro. the extrems. of the other two rt. lines;
and the ⊙ of which ABC is a seg. shall be described.

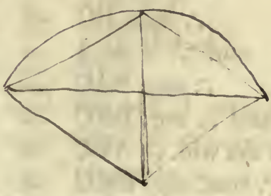
And, if ∠ ABD > ∠ BAD,
it is evident that cent. E shall fall without seg. ABC;
and ∴ seg. ABC < $\frac{1}{2}$ ⊙.

But, if ∠ ABD < ∠ BAD,
then cent. E shall fall within seg. ABC;
and ∴ seg. ABC > $\frac{1}{2}$ ⊙.

Wherefore a segment of a circle being given, &c. &c. &c.

Q. E. F.

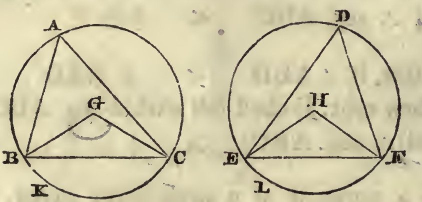




PROP. XXVI. THEOREM.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres, or circumferences.

Let ABC, DEF be equal ⊙s, and the equal ∠s be BGC, CHF at their cents. and ∠s BAC, EDF, at their ○s. Then shall $\overset{\frown}{BKC} = \overset{\frown}{ELF}$.



Join BC, EF;

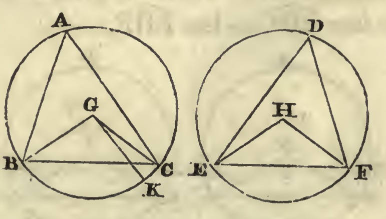
and ∵ ⊙ ABC	=	⊙ DEF,	
∴ BG, GC	=	EH, HF. ea. to ea.	
and ∠ at G	=	∠ at H,	hyp.
∴ base BC	=	base EF;	4. 1.
and ∵ ∠ at A	=	∠ at D,	
∴ seg. BAC	∽	seg. EDF;	11 def. 3.
and ∴ seg. BAC	=	seg. EDF;	29. 3.
but the whl. ⊙ ABC	=	whl. ⊙ DEF	
∴ rem. seg. BKC	=	rem. seg. ELF;	
and ∴ $\overset{\frown}{BKC}$	=	$\overset{\frown}{ELF}$.	

Wherefore in equal circles, &c. &c. Q. E. D.

PROP. XXVII.—Theorem.

In equal circles, the angles which stand upon equal arcs are equal to each other, whether they be at the centres, or circumferences.

Let $\angle$s BGC, EHF at cents. and BAC, EDF at ○s of the equal ⊙s ABC, DEF, stand on the equal arcs BC, EF. Then $\angle$ BGC = $\angle$ EHF, and $\angle$ BAC = $\angle$ EDF.



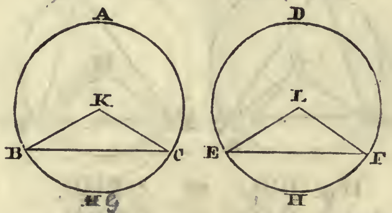
If $\angle$ BGC	$=$	$\angle$ EHF,	
it is plain, that $\angle$ BAC	$=$	$\angle$ EDF;	20. 3.
but assume $\angle$ BGC	$\neq$	$\angle$ EHF,	
then one of them is	$>$	the other;	
let $\angle$ BGC	$>$	$\angle$ EHF;	
& at G, in BG, make $\angle$ BGK	$=$	$\angle$ EHF;	
$\therefore \overset{\frown}{BK}$	$=$	$\overset{\frown}{EF}$;	26. 3.
but $\overset{\frown}{EF}$	$=$	$\overset{\frown}{BC}$,	hyp.
$\therefore \overset{\frown}{BK}$	$=$	$\overset{\frown}{BC}$;	
i. e. less	$=$	greater;	
which is impossible.			
$\therefore \angle$ BGC is not	$\neq$	$\angle$ EHF,	
i. e. $\angle$ BGC	$=$	$\angle$ EHF.	
Now $\angle$ at A	$=$	$\frac{1}{2} \angle$ BGC,	20. 3.
also $\angle$ at D	$=$	$\frac{1}{2} \angle$ EHF,	20. 3.
$\therefore \angle$ BAC	$=$	$\angle$ EDF.	1 ax.

Wherefore in equal circles, &c. &c. Q. E. D.

PROP. XXVIII.—Theorem.

In equal circles, equal right lines cut off equal arcs, the greater equal to the greater, and the less to the less.

Let ABC, DEF be equal ⊙s, and BC, EF equal rt. lines in them, which cut off the two greater arcs BAC, EDF, and the two less BGC, EHF. Then the greater $\overset{\frown}{BAC}$ = greater $\overset{\frown}{EDF}$, and the less $\overset{\frown}{BGC}$ = less $\overset{\frown}{EHF}$.



Take K, L cents. of the ⊙s ; 1.3.
join BK, KC, EL, LF ;
and ∵ ⊙ ABC = ⊙ EDF,
BK, KC = EL, LF, ea. to ea.
and base BC = base EF, hyp.
∴ ∠ BKC = ∠ ELF. 8. 1.
Now ∠s at K and L are at cents. of the ⊙s,
∴ $\overset{\frown}{BGC}$ = $\overset{\frown}{EHF}$; 26. 3.
but whl. ○ ABC = whl. ○ DEF ;
∴ rem. $\overset{\frown}{BAC}$ = rem. $\overset{\frown}{EDF}$. 3 ax.

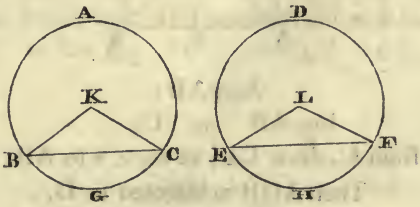
Wherefore in equal circles, equal right lines cut off, &c. &c.

Q. E. D.

PROP. XXIX.—Theorem.

In equal circles, equal arcs are subtended by equal right lines.

Let ABC, DEF be equal ⊙s, and let the arcs BGC, EHF be equal; join BC, EF. Then BC = EF.



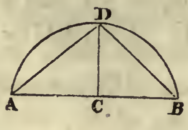
Take K, L cents of the ⊙s;
join BK, KC, EL, LF,
and ∵ $\widehat{BGC}$ = $\widehat{EHF}$,
∴ ∠ BKC = ∠ ELF; 27. 3.
and ∵ ⊙ ABC = ⊙ DEF,
∴ BK, KC = EL, KF ea. to ea.
and they contain equal ∠ s,
∴ base BC = base EF. 4. 1.

Wherefore in equal circles, &c. &c. Q. E. D.

PROP. XXX—Problem.

To bisect a given arc; that is, to divide it into two equal parts.

Let ADB be the given arc; it is required to bisect it.



Join AB;
bis. AB in C; 10. 1.
from C, draw CD, at rt. ∠s to AB;
Then $\widehat{ADB}$ is bisected in D.

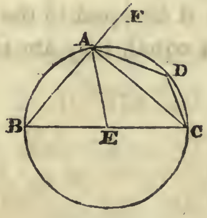
Join AD, DB;
and ∵ AC = CB,
and CD is com. to △s ACD, BCD,
and that ∠ ACD = ∠ BCD,
∴ base AD = base DB;
and ∴ $\widehat{AD}$ = $\widehat{DB}$. 28. 3.

Wherefore $\widehat{ADB}$ is bisected in D. Q. E. F.

PROP. XXXI.—Theorem.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABCD be a ⊙, of which the diam. is BC, and cent. E; and draw CA dividing the circle into the segs. ABC, ADC, and join BA, AD, DC; then the ∠ in the ½ ⊙ BAC is a rt. ∠; and the ∠ in the seg. ABC, which is > ½ ⊙, is < a rt. ∠; and the ∠ in the seg. ADC, which is < ½ ⊙, is > a rt. ∠.



First—Join AE;
prod. BA to F:
and ∵ BE = EA,
∴ ∠ EAB = ∠ ABE: 5. 1.
also ∵ AE = EC,
∴ ∠ EAC = ∠ ACE;
∴ whl. ∠ BAC = ∠ ABC + ∠ ACB:
but in △ ABC; ex. ∠ FAC = ∠s ABC + ACB, 32. 1.
∴ ∠ BAC = ∠ FAC;
and ∴ ea. of the ∠s BAC, FAC = rt. ∠: 10 def. 1.
∴ ∠ BAC in a ½⊙ = rt. ∠.
Secondly ∵ in △ABC; ∠s BAC + ABC < 2 rt. ∠s, 17. 1.
and that ∠ BAC = rt. ∠,
∴ ∠ ABC < rt. ∠;
and ∴ in a seg. > ½ ⊙, the ∠ ABC < rt. ∠.

PROP. XXXI.—CONTINUED.

THIRDLY—∵ ABCD is a quadrilat. fig. in a ⊙,

any two of its opp. ∠s = 2 rt. ∠s;

∴ ∠s ABC + ADC = 2 rt. ∠s; 22.3.

but ∠ ABC < rt. ∠,

∴ ∠ ADC > rt. ∠.

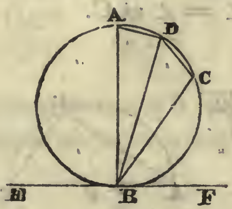
Besides, it is manifest, that the arc of the greater segment ABC falls without the right ∠ CAB; but the arc of the less segment ADC falls within the right ∠ CAF. "And this is "all that is meant, when in the Greek text and the transla-"tions from it, the angle of the greater segment is said to be "greater, and the angle of the less segment is said to be less "than a right ∠.

Cor. From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two, and when the adjacent angles are equal, they are right angles.

PROP. XXXII.—THEOREM.

If a right line touch a circle, and from the point of contact a right line be drawn cutting the circle, the angles which this makes with the line which touches the circle, shall be equal to the angles which are in the alternate segments of the circle.

Let the rt. line EF touch the ⊙ ABCD in B, and from the pt. B let BD be drawn cutting the circle; the ∠s which BD makes with the touching line EF shall be = to the ∠s in the altern. segs. of the ⊙: that is, ∠ FBD = ∠ which is in the seg. DAB, and ∠ DBE = ∠ in the seg. BCD.



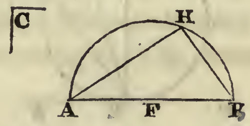
From B, draw BA, at rt. ∠s to EF;
take any pt. C in $\overparen{BD}$;
join AD, DC, CB:
∵ EF touches ⊙ in B,
and that BA is drawn at rt. ∠s to EF from pt. B,
∴ cent. of ⊙ is in AB; 19. 3.
and ∴ ∠ ADB in a ½ ⊙ is a rt. ∠; 31. 3.
and consequently ∠s BAD + ABD = rt. ∠: 32. 1.
but ∠ ABF is rt. ∠,
∴ ∠ ABF = ∠s BAD + ABD;
take away com. ∠ ABD,
∴ rem. ∠ DBF = rem. ∠ BAD;
which ∠ BAD is in the alternate seg. of ⊙.
Again, ∵ ABCD is a quadrilat. fig. in a ⊙,
∴ opp. ∠s BAD + BCD = 2 rt. ∠s; 22. 3.
but ∠s DBF + DBE = 2 rt. ∠s, 13. 1.
∴ ∠s DBF + DBE = ∠s BAD + BCD;
but ∠ DBF = ∠ BAD,
∴ rem. ∠ DBE = rem. ∠ BCD;
which ∠ BCD is in the altern. seg. of ⊙.

Wherefore if a rt. line touch a circle, &c. &c. Q. E. D.

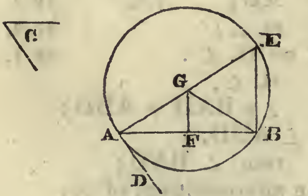
PROP. XXXIII.—Problem.

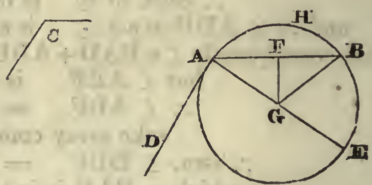
To describe upon a given right line a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

Let AB be the given rt. line, and the ∠ at C the given rectilineal ∠ ; it is required to describe on AB a segment of a ⊙, containing an ∠ = ∠ at C.



First—Let ∠ at C be a rt. ∠.
Bis. AB in F;
with cent. F, and dist. FB, descr. $\frac{1}{2}$ ⊙ AHB;
∴ ∠ AHB in $\frac{1}{2}$ ⊙ = rt. ∠ C. 31. 3.





Secondly—Let C be not a rt. ∠.
At A, in AB, make ∠ BAD = ∠ at C; 23. 1.
from A, draw AE, at rt. ∠s to AD;
bis. AB in F;
from F, draw FG, at rt. ∠s to AB;
join GB.

Then,

PROP. XXXIII. CONTINUED.

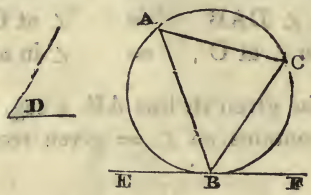
Then, ∵ AF = FB,
and that FG is com. to △s AFG, BFG,
and ∠ AFG = ∠ BFG,
∴ base AG = base GB;
then shall a ⊙ descr. from G, with dist. GA, pass thro. pt. B;
let this ⊙ be AHB:
and ∵ from A, the extremity of diam. AE,
there is drawn AD at rt. ∠s to AE,
∴ AD shall touch ⊙;
and ∵ AB, (drawn from pt. of contact A,) cuts the ⊙,
∴ ∠ DAB = ∠ in altern. seg. AHB;
but ∠ DAB = ∠ at C,
∴ also ∠ at C = ∠ in altern. seg. AHB.

Wherefore on the given rt. line AB, a seg. of a ⊙ has been described which contains an ∠ = given rectilineal ∠ at C. Q. E. F.

PROP. XXXIV.—Problem.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given ⊙, and D the given rectilineal ∠; it is required to cut off a segment from ⊙ ABC that shall contain an ∠ = rectilin. ∠ D.



Draw EF, touching ⊙ in B; 17. 3.

and at B, in BF, make ∠ FBC = ∠ at D: 23. 1.

then, ∵ BC is drawn from pt. of contact B,

∴ ∠ FBC = ∠ in altern. seg. BAC;

but ∠ FBC = ∠ at D,

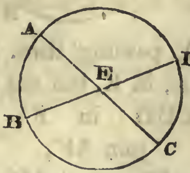
∴ ∠ in altern. seg. BAC = ∠ at D.

∴ A segment BAC is cut from ⊙ ABC containing an ∠ = ∠ at D. Q. E. F.

PROP. XXXV.—Theorem.

If two right lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

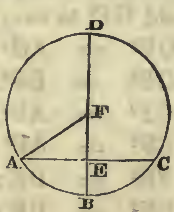
Let AC, BD cut ea. other in pt. E within the $\odot$ ABCD; then shall $AE \times EC = BE \times ED$.



First—Let pt. E be cent. $\odot$;

then since AE, EC, BE, ED = ea. other, 15 def.

it is plain that $AE \times EC = BE \times ED$.



Secondly—Let one of them, BD, pass thro. cent. and cut the other AC, which does not pass thro. cent. at rt. $\angle$s in the pt. E,

then BD bisects AC. 3. 3.

Bisect BD in F;

and F is cent. of $\odot$ ABCD:

join AF:

and $\because$ BD bisects AC,

$\therefore$ AE = EC;

and $\because$ BD is bisected in F,

and that BD is also divided into two $\neq$ parts in E,

$\therefore BE \times ED + EF^2 = FB^2$ i. e. FA^2; 5. 2.

but $AE^2 + EF^2 = FA^2$,

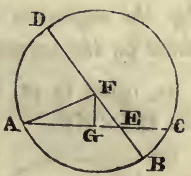
$\therefore BE \times ED + EF^2 = AE^2 + EF^2$;

take away com. EF^2,

$\therefore$ rem. $BE \times ED$ = rem. AE^2 i.e. $AE \times EC$. 3 ax.

Thirdly,

PROP. XXXV. CONTINUED.



THIRDLY—Let BD, passing thro. cent., cut AC, which does not pass thro. cent., in E, but not at rt. ∠s.

Bisect BD in F;

join AF;

from F, draw FG ⊥ AC;

$\therefore$ AG = GC; 3. 3.

and $\therefore AE \times EC + GE^2 = AG^2$; 5. 2.

add GF^2 to ea.;

$\therefore AE \times EC + EG^2 + GF^2 = AG^2 + GF^2$; 2 ax.

but $EG^2 + GF^2 = EF^2$,

and also $AG^2 + GF^2 = AF^2$,

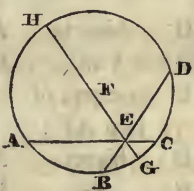
$\therefore AE \times EC + EF^2 = AF^2$ i. e. FB^2;

but $FB^2 = BE \times ED + EF^2$, 5. 2.

$\therefore AE \times EC + EF^2 = BE \times ED + EF^2$;

take away com. EF^2,

$\therefore$ rem. $AE \times EC$ = rem. $BE \times ED$; 3 ax.



LASTLY—Let neither AC or BD pass thro. cent. of ⊙.

Take F cent. of ⊙; 1. 3.

through E, draw Dia. GEFH:

then, ∵ $AE \times EC = GE \times EH$,* 3d case.

and that similarly $BE \times ED = GE \times EH$,

$\therefore AE \times EC = BE \times ED$. 1 ax.

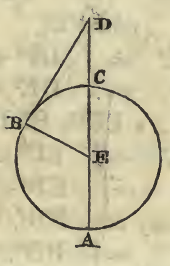
Wherefore if two rt. lines within a circle, &c. &c. Q. E. D.

* That is, by substituting HG for DB in the last fig.

PROP. XXXVI.—THEOREM.

If from any point without a circle two right lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

Let D be any pt. without ⊙ ABC, and DCA, BD two rt. lines drawn from it, of which DCA cuts the ⊙ and DB touches the same. Then shall $AD \times DC = BD^2$.



Either DCA passes thro. cent. or it does not.

FIRST.—Let DCA pass. thro. cent. E.

Join EB;

∴ ∠ EBD is a rt. ∠; 18. 3.

and ∵ AC is bisected in E and produced to D,

∴ $AD \times DC + EC^2 = ED^2$; 6. 2.

but $EC^2 = EB^2$,

(for EC = EB,)

also $ED^2 = EB^2 + BD^2$, 47. 1.

(for ∠ EBD is a rt. ∠,)

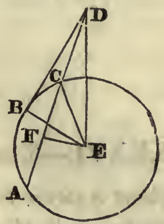
∴ $AD \times DC + EB^2 = EB^2 + BD^2$;

take away com. EB^2,

∴ rem. $AD \times DC = BD^2$.

SECONDLY,

PROP. XXXVI.—CONTINUED.



SECONDLY.—Let DCA not pass thro. cent. ⊙.

Take E cent of ⊙;

draw EF ⊥ AC;

join ED, EC, EB;

then, ∵ EF is rt. ∠s to AC;

∴ AF = FC; 3. 3.

and ∵ AC is bisected in F and produced to D,

$\therefore AD \times DC + FC^2 = FD^2$; 6. 2.

add FE^2,

$\therefore AD \times DC + CF^2 + FE^2 = DF^2 + FE^2$;

but $DF^2 + FE^2 = DE^2$, i.e. $EB^2 + BD^2$, 47. 1.

(for ea. of the ∠s EFD, EBD is a rt. ∠,)

&similarly also $CF^2 + FE^2 = EC^2$, i. e. EB^2, 47. 1. and 15 def. 1.

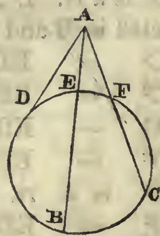
$\therefore AD \times DC + EB^2 = EB^2 + BD^2$;

take away com. EB^2,

∴ rem. $AD \times DC = BD^2$,

Wherefore, if from a pt. &c. &c. Q. E. D.

Cor. If from a point without a circle two right lines as AB, AC be drawn cutting the circle, then $AB \times AE = AC \times AF$.



for ∵ $BA \times AE = AD^2$,

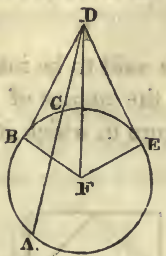
and also $AC \times AF = AD^2$,

$\therefore AB \times AE = AC \times AF$. 1 ax.

PROP. XXXVII.—Theorem.

If from a point without a circle there be drawn two right lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the square of the line which meets it, the line which meets shall touch the circle.

Let any pt. D be taken without the ⊙ ABC, and from it let two rt. lines DCA, DB, be drawn, of which, DCA cuts the ⊙, and DB meets it; if $AD \times DC = DB^2$, then DB touches the ⊙.



Draw DE touching ⊙ ABC in E; 17. 3.
find F cent. ⊙; 1. 3.
join FB, FD, EE;
then ∠ FED is a rt. ∠: 18. 3.
and ∵ DE touches ⊙ ABC,
and that DCA cuts ⊙ ABC,
∴ $AD \times DC = DE^2$; 36. 3.
but $AD \times DC = DB^2$; hyp.
∴ $DB^2 = DE^2$;
and ∴ DB = DE;
and ∵ also FB = FE,
then DB, BF = DE, EF ea. to ea.,
and ∵ base DF is com. to △ s DFB, DFE,
∴ ∠ DEF = ∠ DBF; 8. 1.
but ∠ DEF is a rt. ∠,
∴ ∠ DBF is a rt. ∠;
and ∴ DB is at rt. ∠ s BF;
but BF produced is diam.
∴ DB touches ⊙ ABC. 16. 3.

Wherefore if from a point from without a circle, &c. &c. Q.E.D.

BOOK IV.

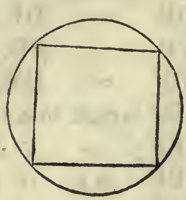
DEFINITIONS.

I.

A rectilineal figure is said to be inscribed in another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

II.

In like manner, a figure is said to be described about another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.



III.

A rectilineal figure is said to be inscribed in a circle, when all the angles of the inscribed figure are upon the circumference of the circle.

IV.

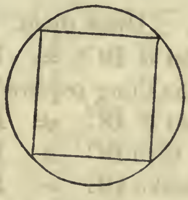
A rectilineal figure is said to be described about a circle, when each side of the circumscribed figure touches the circumference of the circle.

V.

In like manner, a circle is said to be inscribed in a rectilineal figure, when the circumference of the circle touches each side of the figure.

VI.

A circle is said to be described about a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.



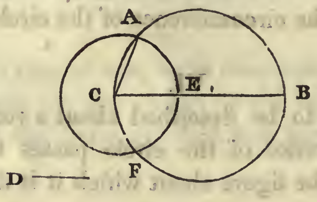
VII.

A right line is said to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I.—PROBLEM.

In a given circle to place a right line, equal to a given right line not greater than the diameter of the circle.

Let ABC be the given ⊙ and D the given rt. line; it is required to place in the ⊙ ABC a rt. line = D which is not > diam. of ⊙.



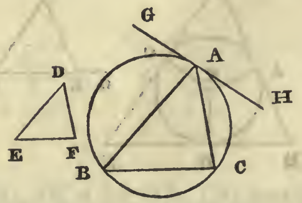
Draw diam. BC;
and if BC = D,
the thing required is done.
But if BC ≠ D,
then BC > D;
make CE = D; 3. 1.
and with cent. C, and dist. CE descr. ⊙ AEF;
then ∵ C is cent. ⊙ AEF,
∴ AC = CE;
but CE = D,
∴ AC = D.

∴ In the given ⊙ ABC is placed a rt. line AC = D not > diameter. Q. E. F.

PROP. II.—Problem.

In a given circle to inscribe a triangle equiangular to a given triangle.

Let ABC be the given ⊙, and DEF the given △; it is required to inscribe in the ⊙ ABC a △ equiang. to △ DEF.



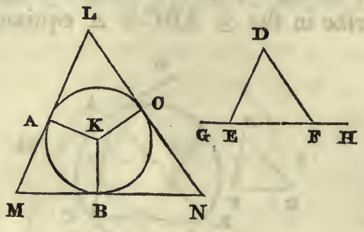
Draw GH touching ⊙ in A;
at A, in AH, make ∠ HAC = ∠ DEF; } 23. 1.
and at A, in GA, make ∠ GAB = ∠ DFE; }
join BC;
then, ∵ GH touches ⊙ ABC,
and that AC is drawn from pt. of contact A,
∴ ∠ HAC = ∠ ABC; 32. 3.
but ∠ HAC = ∠ DEF,
∴ ∠ ABC = ∠ DEF;
similarly ∠ ACB = ∠ DFE,
∴ rem. ∠ BAC = rem. ∠ EDF;
∴ △ ABC is equiang. to △ DEF.

Therefore in the given ⊙ ABC has been inscribed a △ ABC equiang. to △ DEF. Q. E. F.

PROP. III.—Problem.

About a given circle to describe a triangle equiangular to a given triangle.

Let ABC be the given ⊙, and DEF the given △; it is required to describe about the ⊙ ABC a △ equiang. to △ DEF.



Produce EF both ways to G and H;

find K cent. ⊙ ABC; 1. 3.

from K, draw KB, to ⊙;

at K, in KB, make ∠ AKB = ∠ DEG; }

and also ∠ BKC = ∠ DFH; } 23. 1.

thro. A, B, C, draw LM, MN, NL, touching ⊙ ABC:

and ∴ all the ∠s at, A, B, C are rt. ∠s; 18. 3.

and ∵ 4 ∠s of fig. AMKB = 4 rt. ∠s,

(for fig. AMKB can be ÷ into two △s,)

and that ∠s KAM, MBK are 2 rt. ∠s,

∴ ∠s AMB + AKB = 2 rt. ∠s;

but ∠s DEG + DEF = 2 rt. ∠s, 13. 1.

∴ ∠s AMB + AKB = ∠s DEG + DEF;

but by constr. ∠ AKB = ∠ DEG,

∴ rem. ∠ AMB = rem. ∠ DEF;

similarly ∠ LNM = ∠ DFE,

∴ rem. ∠ MLN = rem. ∠ EDF; 32. 1.

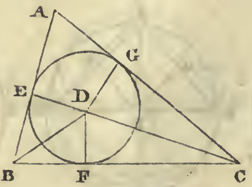
and ∴ △ MLN is equiang. to △ DEF.

Wherefore about given ⊙ ABC has been described a △ equiang. to △ DEF. Q. E. F.

PROP. IV. Problem.

To inscribe a circle in a given triangle.

Let ABC be the given △; it is required to inscribe a ⊙ in ABC.



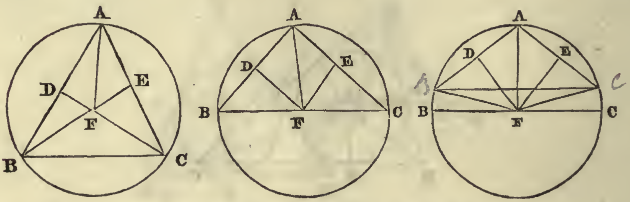
Bisect ∠s ABC, BCA by BD, CD meeting in D; 9.1.
from D, draw DE, DF, DG ⊥ AB, BC, CA; 12.1.
and ∵ ∠EBD = ∠DBF,
and that rt. ∠ BED = rt. ∠ BFD,
then ∠s DBE, BED = ∠s DBF, BFD ea. to ea.;
and ∵ BD is com. and opposite,
∴ DE = DF: 26.1.
similarly DG = DF,
∴ DE, DF, DG = ea. other.
∴ with cent. D and dist. DE, DF, or DG descr. ⊙ EFG;
and ∵ ∠s at E, F, and G are rt. ∠s,
∴ ⊙ EFG shall touch the sides AB, BC, CA; 16.3.
∴ ea. of AB, BC, CA touches ⊙ EFG;
and ∴ ⊙ EFG is inscribed in △ ABC.

Q. E. F.

PROP. V.—Problem.

To describe a circle about a given triangle.

Let ABC be the given △; it is required to describe a ⊙ about △ ABC.



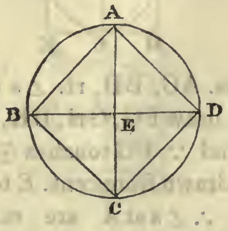
Bis. AB, AC in D and E;
draw DF and EF at rt. ∠ s to AB, AC;
then shall DF, EF meet in F;
for, if they do not meet,
then DF ∥ EF,
and ∴ also AB ∥ AC;
which is absurd;
let DF, EF meet in F,
and, if F is not in BC,
join BF, FC;
and, ∵ AD = DB,
and that DF is com. to △ s ADF, BDF,
and that rt. ∠ ADF = rt. ∠ BDF,
∴ BF = AF; 4. 1.
similarly CF = AF;
and ∴ BF = CF; 1 ax.
∴ AF, BF, CF = ea. other.

Therefore a ⊙ described with cent. F and dist. any one of them will pass thro. extrems. of the other two, and be described about △ ABC. Q. E. F.

PROP. VI.—Problem.

To inscribe a square in a given circle.

Let ABCD be the given ⊙ ; it is required to inscribe a sq. in ⊙ ABCD.



Draw diams. AC, BD at rt. ∠s to ea. other ;
join AB, BC, CD, DA ;
and ∵ BE = ED,
(for E is cent. of ⊙,)
and that AE is com.
and rt. ∠ BEA = rt. ∠ AED,
∴ base AB = base AD ; 4. 1.
similarly BC, CD = BA, or AD ;
∴ AB, BC, CD, DA = ea. other ;
and ∴ fig. ABCD is equilat.
Again, ∵ BAD is $\frac{1}{2}$ ⊙,
∴ ∠ BAD = rt. ∠ ; 31. 3.
simil. ∠ ADC, ∠ DCB, or ∠ CBA = rt. ∠,
∴ fig. ABCD is also equiang.
and ∴ ABCD is a square.

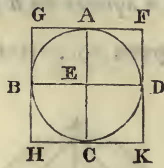
Therefore, in given ⊙ ABCD, has been inscribed a square.

Q. E. F.

PROP. VII.—Problem.

To describe a square about a given circle.

Let ABCD be the given ⊙. It is required to describe a square about it.



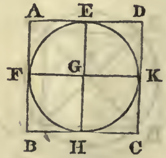
Draw diams. AC, BD, rt. ∠s to ea. other,
and thro. A, B, C, D, draw FG, GH, HK, KF touching ⊙ : 17. 3.
and ∵ FG touches ⊙ ABCD,
and, that EA is drawn from cent. E to pt. of contact A,
∴ ∠s at A are rt. ∠s ;
similarly ∠s at B, C, D are rt. ∠s : 18. 3.
and ∵ ∠ AEB is a rt. ∠,
and that ∠ EBG is a rt. ∠,
∴ GH ∥ AC ; 28. 1.
similarly AC ∥ FK ;
and GF or HK ∥ BD ;
∴ figs. GK, GC, AK, FB, BK are ▭s ;
∴ GH = FK, 34. 1.
and GF = HK ;
and ∵ AC = BD,
and that AC = GH or FK,
and BD = GF or HK,
∴ ea. of GH, FK = GF or HK ;
∴ quadrilat. fig. GK is equilat.
Again, ∵ fig. GE is a ▭,
and that ∠ BEA is a rt. ∠,
∴ ∠ AGB is a rt. ∠ ; 34. 1.
similarly ∠s GHK, HKF, KFG are rt. ∠s ;
and ∴ fig. GK is equiang.
and ∴ GK is a square.
And it is described about the ⊙ ABCD.

Q. E. F.

PROP. VIII.—Problem.

To inscribe a circle in a given square.

Let ABCD be the given square, required to inscribe a $\odot$ in it.



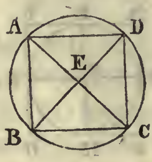
Bisect AB, AD in F and E;
thro. E, draw EH $\parallel$ AB or DC;
and thro. F, draw FK $\parallel$ AD or BC;
$\therefore$ figs. AK, KB, AH, HD, AG, GC, BG, GD are $\square$ s:
and $\therefore$ their opp. sides = ea. other:
and $\because$ AD = AB,
and AE = $\frac{1}{2}$ AD,
and that AF = $\frac{1}{2}$ AB,
$\therefore$ AE = AF;
and $\therefore$ FG = GE: 34. 1.
similarly ea. of GH, GK = FG or GE;
$\therefore$ GE, GF, GH, GK = ea. other;
and $\therefore$ a $\odot$, described from cent. G, with dist. any one of them, shall pass thro. extrems. of the other three, and touch the sides AB, BC, CD, DA:
and $\because$ $\angle$ s at E, F, H, K are rt. $\angle$ s, 29. 1.
$\therefore$ AB, BC, CD, DA are at rt. $\angle$ s to diams. EH, FK;
and $\therefore$ AB, BC, CD, DA touch $\odot$ EFHK; 16. 3.
and therefore $\odot$ EFHK is inscribed in given sq. ABCD.

Q. E. F.

PROP. IX.—Problem.

To describe a circle about a given square.

Let ABCD be the given sq. It is required to describe a ⊙ about it.



Join AC, BD, cutting ea. other in E.

Then, ∵ AD = AB,

and AC is com.,

and that base BC = base DC,

∴ ∠ DAC = ∠ BAC; 8. 1.

and ∴ ∠ DAB is bis. by AC:

similarly, ∠ s ABC, BCD and CDA are bis. by BD and AC:

and ∵ ∠ DAB = ∠ ABC,

and ∠ EAB = $\frac{1}{2}$ ∠ DAB,

and that ∠ EBA = $\frac{1}{2}$ ∠ ABC,

∴ ∠ EAB = EBA;

∴ EA = EB: 6. 1.

similarly, ea. of EC, ED = EA or EB:

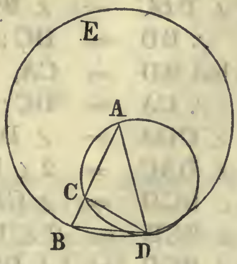
∴ EA, EB, EC and ED = ea. other:

and therefore a ⊙ described from cent. E and dist. any one of them shall pass thro. extrems. of the other three, and be described about a given sq. ABCD.

Q. E. F.

PROP. X.—Problem.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.



Take any rt. line AB ;
divide AB in C,
so that, AB × BC = AC²; 11. 2.
with cent. A, and dist. AB descr. ⊙ BDE ;
in ⊙ BDE place a rt. line BD = AC, ≯ dia. of ⊙;
join DA, DC ;
about △ ACD descr. ⊙ ACD :
then △ ABD is such as was required ;
i.e. ea. of ∠s ABD, BDA = 2 ∠ BAD.
For, ∵ AB × BC = AC²,
and that AC = BD,
∴ AB × BC = BD²;
and ∵ from B, without ⊙ ACD ; BCA, BD are drawn to the ○,
of which BCA cuts the ⊙,
and BD meets ⊙,
and that AB × BC = BD²,
∴ BD touches ⊙ ACD : 37. 3.
and ∵ also, DC is drawn from D the pt. of contact,
∴ ∠ BDC = ∠ DAC in altern. seg. 32. 3.
add ∠ CDA,
∴ whl. ∠ BDA = ∠ CDA + ∠ DAC ;
but

PROP. X. CONTINUED.

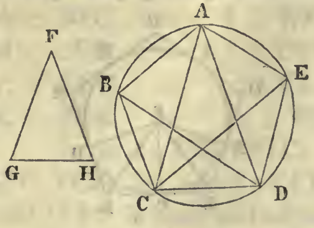
but the ex. $\angle$ BCD $=$ $\angle$s CDA $+$ DAC, 32. 1.
$\therefore$ $\angle$ BDA $=$ $\angle$ BCD;
but $\angle$ BDA $=$ $\angle$ CBD, 5. 1.
(for AD $=$ AB,)
$\therefore$ $\angle$ CBD or $\angle$ BDA $=$ $\angle$ BCD;
and $\therefore$ $\angle$s BDA, DBA, & BCD $=$ ea. other:
and $\because$ $\angle$ DBC $=$ $\angle$ BCD,
$\therefore$ BD $=$ DC; 6. 1.
but BD $=$ CA,
$\therefore$ CA $=$ DC;
and $\therefore$ $\angle$ CDA $=$ $\angle$ DAC; 5. 1.
$\therefore$ $\angle$ CDA $+$ $\angle$ DAC $=$ 2 $\angle$ DAC:
but $\angle$ BCD $=$ $\angle$ CDA $+$ DAC,
$\therefore$ $\angle$ BCD $=$ 2 $\angle$ DAC;
and $\angle$ BCD $=$ $\angle$ BDA or $\angle$ DBA,
$\therefore$ ea. of $\angle$s BDA, DBA $=$ 2 $\angle$ DAB.

Wherefore an isosceles $\triangle$ is described having ea. of its $\angle$s at the base $=$ twice $\angle$ at vertex. Q. E. F.

PROP. XI.—Problem.

To inscribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given ⊙; it is required to inscribe in it an equilat. and equiang. pentagon.



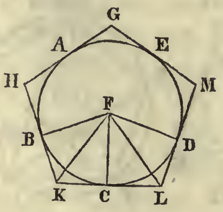
Descr. an isosceles △ FGH,
having ea. of its ∠ s FGH, GHF = 2 ∠ GFH ; 10. 4.
and inscr. in ⊙ ABCDE, a △ ACD equiang. to △ FGH,
so that ∠ CAD = ∠ at F, 2. 4.
and ea. of the ∠ s ACD, CDA = ∠ at G or ∠ at H;
and ∴ ea. of the ∠ s ACD, CDA = 2 ∠ CAD :
bisect ∠ s ACD, CDA by CE, DB ; 9. 1.
join AB, BC, CD, DE, EA :
then fig. ABCDE is the required ptgon.
∵ ea. of the ∠ s ACD, CDA = 2 ∠ CAD,
and that they are bisected by CE, DB,
∴ the 5 ∠ s { DAC, ACE, ECD, CDB and BDA } = ea. other :
and ∵ equal ∠ s stand on equal arcs, 26. 3.
∴ $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, $\widehat{DE}$, $\widehat{EA}$ = ea. other ;
and ∴ AB, BC, CD, DE, EA = ea. other ; 29. 3.
∴ ptgon. ABCDE is equilat.
Again, ∵ $\widehat{AB}$ = $\widehat{DE}$
add $\widehat{BCD}$,
∴ whl. $\widehat{ABD}$ = whl. $\widehat{EDB}$;
and ∵ ∠ AED stands on $\widehat{ABD}$,
and that ∠ BAE stands on $\widehat{EDB}$,
∴ ∠ BAE = ∠ AED : 27. 3.
simi. ea. of ∠ s ABC, BCD, CDE = ∠ BAE or ∠ AED :
∴ ptgon. ABCDE is also equiang.

Wherefore in given ⊙ ABCDE has been inscribed an equilat. and equiang. pentagon. Q. E. F.

PROP. XII.—Problem.

To describe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given $\odot$; it is required to describe about it an equilat. and equiang. pentagon.



Let $\angle$ s of a ptgon. inscribed in the $\odot$ be in pts. A, B, C, D, E,
and so that $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, $\widehat{DE}$, $\widehat{EA}$ = ea. other; 11. 4.
thro. A, B, C, D, E draw GH, HK, KL, LM, MG touching $\odot$;
take F cent. $\odot$;
join FB, FK, FC, FL, FD:
and $\because$ KL touches $\odot$ in C,
and that FC is drawn from F to pt. of contact C,

$\therefore$ FC $\perp$ KL; 18. 3.
$\therefore$ ea. of the $\angle$ s at C is a rt. $\angle$:
similarly the $\angle$ s at B and D are rt. $\angle$ s:
and $\because$ $\angle$ FCK is a rt. $\angle$,
$\therefore$ $FK^2 = FC^2 + CK^2$; 47. 1.
similarly $FK^2 = FB^2 + BK^2$:
and $\therefore$ $FC^2 + CK^2 = FB^2 + BK^2$, 1 ax.
of which $FC^2 = FB^2$,
(for FC = FB)
$\therefore$ $CK^2 = BK^2$;
and $\therefore$ CK = BK:
and $\because$ FB = FC,
and FK com. to $\triangle$ s FBK, FCK,
and that base CK = base BK,
$\therefore$ $\angle$ BFK = $\angle$ KFC, }
and $\angle$ BKF = $\angle$ FKC; } 8. 1.
$\therefore$ $\angle$ BFC = 2 $\angle$ KFC,
and $\angle$ BKC = 2 $\angle$ FKC:

similarly,

PROP. XII. CONTINUED.

similarly, { ∠ CFD	=	2 ∠ CFL,	
and ∠ CLD	=	2 ∠ CLF. }	
Again, ∵ $\widehat{BC}$	=	$\widehat{CD}$,	
∴ ∠ BFC	=	∠ CFD :	27.3.
now ∠ BFC	=	2 ∠ KFC,	
and ∠ CFD	=	2 ∠ CFL,	
∴ ∠ KFC	=	∠ CFL ;	
and ∵ also rt. ∠ FCK	=	rt. ∠ FCL,	
∴, in △s FKC, FLC,			
are two ∠s KFC, FCK	=	two ∠s CFL, FCL ea. to ea. :	
and ∵ FC is com. and adjacent to = ∠s,			
∴ KC	=	CL ; }	26. 1.
and ∠ FKC	=	∠ FLC : }	
and ∵ KC	=	CL,	
∴ KL	=	2 KC :	
similarly, HK	=	2 BK :	
and ∵ BK	=	KC,	demon.
and KL	=	2 KC,	
and that HK	=	2 BK,	
∴ HK	=	KL :	
similarly, ea. of GH, GM, ML	=	HK or KL :	
∴ ptgon. GHKLM	is	equilat.	
Again, ∵ ∠ FKC	=	∠ FLC,	
and ∠ HKL	=	2 ∠ FKC,	
and that ∠ KLM	=	2 ∠ FLC,	
∴ ∠ HKL	=	∠ KLM :	
similarly ea. of ∠s KHG, HGM, GML	=	∠ HKL or ∠ KLM :	
∴ ptgon. GHKLM	is also	equiang.	

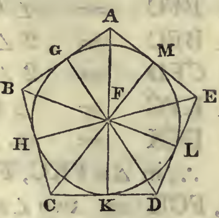
and it is described about the given ⊙ ABCDE.

Q. E. F.

PROP. XIII.—Problem.

To inscribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilat. and equiang. pentagon; it is required to inscribe a ⊙ in it.



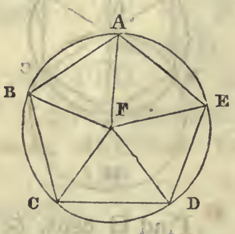
Bisect ∠s BCD, CDE by CF, DF; 9. 1.
from F, where they meet, draw FB, FA, FE:
then ∵ BC = CD,
and CF com. to △s BCF, DCF,
and that ∠ BCF = ∠ DCF,
∴ base BF = base FD, }
and ∠ CBF = ∠ CDF: } 4. 1.
and, ∵ ∠ CDE = 2 ∠ CDF,
and that ∠ CDE = ∠ CBA,
and ∠ CDF = ∠ CBF,
∴ ∠ CBA = 2 ∠ CBF;
and ∴ ∠ ABF = ∠ CBF;
and consequently ∠ ABC is bis. by BF:
similarly ∠s BAE, AED are bis. by AF, FE:
from F, draw { FG, FH ⊥ AB, BC, / FK, FL ⊥ CD, DE, / and FM ⊥ AE, } respectively;
and ∵ ∠ HCF = ∠ KCF,
and rt. ∠ FHC = rt. ∠ FKC,
then in the △s FHC, FKC,
are two ∠s FHC, HCF = two ∠s FKC, KCF ea. to ea.;
and ∵ FC is com. and oppos. to = ∠s,
∴ FH = FK: 26. 1.
similarly ea. of FL, FM, FG = FH or FK:
∴ the five rt. lines = ea. other.
Therefore a ⊙ described from F with dist. any one of them, shall pass thro. the extrems. of the other four, and touch the sides AB, BC, CD, DE, EA.
And ∵ ∠s at pts. G, H, K, L, M are rt. ∠s,
∴ AB, BC, CD, DE, EA touch ⊙ so described. 26. 3.
Therefore a ⊙ has been inscribed in the given ptgon. ABCDE.

Q. E. F.

PROP. XIV.—Problem.

To describe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the equilat. and equiang. ptgon. Required to describe a ⊙ about it.



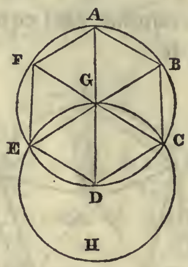
Bis. ∠s BCD, CDE by CF, DF meeting in F;
from F, draw FB, FA, FE to pts. B, A, E.
And it may be shewn as in the preceding proposition;
that FA, FB, FE bis. ∠s CBA, BAE, AED:
and ∵ ∠ BCD = ∠ CDE,
and that ∠ FCD = $\frac{1}{2}$ ∠ BCD,
and ∠ CDF = $\frac{1}{2}$ ∠ CDE,
∴ ∠ FCD = ∠ CDF;
∴ FC = FD: 6. 1.
similarly FB, FA, or FE = FC, or FD:
∴ the five rt. lines = ea. other.

Therefore a ⊙ described from cent. F, with dist. any one of them shall pass thro. the pts. A, B, C, D, E, and be described about the ptgon. ABCDE. Q. E. F.

PROP. XV.—PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given ⊙; required to inscribe an equilat. and equian. hxgon. in it.



Find G cent. ⊙;
draw dia. AGD;
with cent. D, and dist. DG, descr. ⊙ EGCH;
join EG, GC;
produce EG, CG, to B and F;
join AB, BC, CD, DE, EF, FA:
the hxgon. ABCDEF is equilat. and equiang.

For ∵ G is cent. ⊙ ABCDEF,
∴ GE = GD;
and ∵ D is cent. ⊙ EGCH,
∴ DE = DG;
∴ GE = DE; 1 ax.
and ∴ △ EGD is equilat.
and its ∠s EGD, GDE, DEG = ea. other: 5. 1.
and ∵ three ∠s of a △ = 2 rt ∠s, 32. 1.
∴ ∠ EGD = $\frac{1}{3}$ of 2 rt. ∠s:
similarly ∠ DGC = $\frac{1}{3}$ of 2 rt. ∠s:
and ∵ CG stands on EB,
and makes adj. ∠s EGC, CGB = 2 rt. ∠s 13. 1.
∴ rem. ∠ CGB = $\frac{1}{3}$ of 2 rt. ∠s;
∴ ∠s EGD, DGC, CGB = each other;

also

PROP. XV.—CONTINUED.

also vert. ∠s BGA, AGF, FGE = ∠s EGD, DGC, CGB, [ea. to ea. 15. 1.

∴ the six ∠s = ea. other;

and ∴ $\left\{ \begin{array}{l} \overset{\frown}{AB},\ \overset{\frown}{BC},\ \overset{\frown}{CD}, \\ \overset{\frown}{DE},\ \overset{\frown}{EF},\ \overset{\frown}{FA}, \end{array} \right\}$ = ea. other; 26. 3.

and ∴ $\left\{ \begin{array}{l} AB,\ BC,\ CD, \\ DE,\ EF,\ FA, \end{array} \right\}$ = ea. other; 9. 3.

and ∴ hxgon. ABCDEF is equilat.

Again ∵ $\overset{\frown}{AF}$ = $\overset{\frown}{ED}$,

add $\overset{\frown}{ACD}$,

∴ whl. $\overset{\frown}{FBD}$ = whl. $\overset{\frown}{ECA}$;

and ∵ ∠ FED stands on $\overset{\frown}{FBD}$,

and ∠ AFE on $\overset{\frown}{ECA}$,

∴ ∠ AFE = ∠ FED;

similarly ea. of the other four ∠s = ∠ AFE, or ∠ FED:

and ∴ the six ∠s = ea. other:

∴ hxgon. ABCDEF is also equiang.

Therefore an equilat. and equiang. hexagon. has been inscribed in given ⊙. Q. E. F.

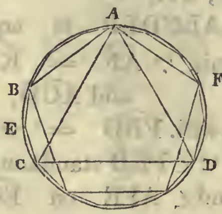
Cor. From this it is manifest, that the side of the hexagon is equal to the right line from the centre, that is to the semidiameter of the circle.

And if through the points A, B, C, D, E, F, there be drawn right lines touching the circle, an equilateral, and equiangular hexagon shall be described about it, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method like that used for the pentagon.

PROP. XVI—Problem.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given ⊙ ; required to inscribe an equilat. and equiang. quindecagon in it.



In ⊙ ABCD inscr. an equilat. △ ACD ; 2. 4.
and also, in same ⊙, inscr. an equilat. and equiang. ptgon ; 11. 4.
then $\widehat{ABC}$ = $\frac{1}{3}$ of whl. ○ :
and $\widehat{AB}$ = $\frac{1}{5}$ of whl. ○ :
and consequently, if whl. ○ contain 15 equal parts,
then $\widehat{ABC}$ contains 5 such parts,
and $\widehat{AB}$ contains 3 such parts ;
and ∴ their difference $\widehat{BC}$ contains 2 such parts :
now bis. $\widehat{BC}$ in E, 30. 3.
and ∴ $\widehat{BE}$, or $\widehat{EC}$ will contain 1 such part.

And consequently if BE, or EC be drawn, and their equals extended round the whl. ⊙ ; an equilat. and equiang. quindecagon shall be inscribed in it. Q. E. F.

And in the same manner as was done in the pentagon, if, through the point of division made by inscribing the quindecagon, right lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it ; and likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

BOOK V

DEFINITIONS.

I.

A less magnitude is said to be a part of a greater magnitude when the less measures the greater; that is, 'when the 'less is contained a certain number of times exactly in the 'greater.'

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less, that is, 'when the 'greater contains the less a certain number of times exactly.'

III.

" Ratio is a mutual relation of two magnitudes of the same " kind to one another, in respect of quantity."

IV.

Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other.

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth: or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth: or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

VI.

Magnitudes which have the same ratio are called proportionals. 'N.B. When four magnitudes are proportionals, 'it is usually expressed by saying, the first is to the second, as 'the third to the fourth.'

VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

VIII.

"Analogy or proportion, is the similitude of ratios."

IX.

Proportion consists in three terms at least.

X.

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c. increasing the denomination still by unity, in any number of proportionals.

Definition A, to wit, of compound ratio.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to

C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F; and B to C the same ratio that G has to H; and C to D the same that K has to L; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

XII.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

'Geometers make use of the following technical words, to 'signify certain ways of changing either the order or magni- 'tude of proportionals, so that they continue still to be pro- 'portionals.'

XIII.

Permutando, or alternando, by permutation or alternately. This word is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth: as is shown in the 16th Prop. of this fifth book.

XIV.

Invertendo, by inversion; when there are four proportionals, and it is inferred, that the second is to the first as the fourth to the third. Prop. B. Book 5.

XV.

Componendo, by composition; when there are four proportionals, and it is inferred, that the first together with the second, is to the second, as the third together with the fourth, is to the fourth. 18th Prop. Book 5.

XVI.

Dividendo, by division; when there are four proportionals, and it is inferred, that the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth. 17th Prop. Book 5.

XVII.

Convertendo, by conversion; when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third to its excess above the fourth. Prop. E. Book 5.

XVIII.

Ex æquali (sc. distantiâ), or ex æquo, from equality of distance: when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred, that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: 'Of this there are the two 'following kinds, which arise from the different order in which 'the magnitudes are taken, two and two.'

XIX.

Ex æquali, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonstrated in the 22nd Prop. Book 5.

XX.

Ex æquali in proportione perturbatâ seu inordinatâ, from equality in perturbate or disorderly proportion.* This term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two, to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from

* 4 Prop. lib. 2, Archemedis de sphærâ et cylindro.

the last to the last but two of the second rank; and so on in a cross order: and the inference is in the 18th definition. It is demonstrated in 23 Prop. Book 5.

AXIOMS.

I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

II.

Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

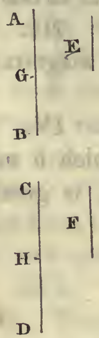
IV.

That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROP. I.—Theorem.

If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall the first magnitude be of all the other.

Let any No. of mags. AB, CD be equimults. of as many others, E, F, ea. of ea.; then shall AB + CD be same mult. of E + F, that AB is of E.



∵ AB is same mult. of E, that CD is of F,

∴ (No. mags. in AB which = E) = (No. mags. in CD which = F).

Divide AB into mags. AG, GB ea. = E;

and CD into mags. CH, HD ea. = F;

then No. mags. CH, HD = No. mags. AG, GB;

and ∵ AG = E,

and CH = F,

∴ AG + CH = E + F: 2 ax.

similarly, GB + HD = E + F:

∴ (No. mags. in AB which = E) = (No. mags. in AB + CD which = E + F);

∴ whatever mult. AB is of E, the same is AB + CD of E + F.

Therefore, if any number of magnitudes, &c. &c.

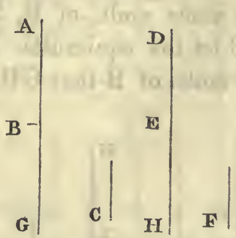
"For the same demonstration holds in any number of magnitudes, which is here applied to two."

Q. E. D.

PROP. II.—Theorem.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second, that the third together with the sixth is of the fourth.

Let AB the 1st be the same mult. of C the 2d that DE the 3d is of F the 4th.; also BG the 5th the same mult. of C the 2d that EH the 6th is of F the 4th. Then is AG, (the 1st + the 5th,) the same mult. of C that DH, (the 3d + the 6th,) is of F.



∵ AB is same mult. of C that DE is of F,

∴ (No. mags. in AB which = C) = (No. mags. in DE which = F):

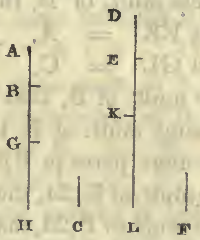
similarly, (No. mags. in BG which = C) = (No. mags. in EH which = F):

∴ (No. mags. in whl. AG which = C) = (No. mags. in whl. DH which = F);

∴ AG is same mult. of C, that DH is of F;

i. e. AG, 1st + 5th, is same mult. of C, 2d, that DH, 3d + 6th, is of F, 4th.

If therefore, the first be the same multiple, &c. &c. Q. E. D.

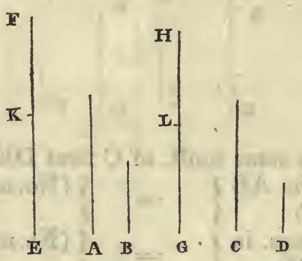


Cor. "From this it is plain, that if any number of magnitudes AB, BG, GH, be equimultiples of another C; and as many DE, EK, KL, be the same multiples of F, each of each; the whole of the first, viz. AH is the same multiple of C that the whole of the last, viz. DL is of F."

PROP. III.—Theorem.

If the first be the same multiple of the second, which the third is of the fourth; and if of the first and third there be taken equimultiples, these shall be equimultiples, the one of the second, and the other of the fourth.

Let A, 1st, be the same mult. of B, 2d, that C, 3d, is of D, 4th; and of A, C let the equimults. EF, GH be taken: then EF is the same mult. of B that GH is of D.



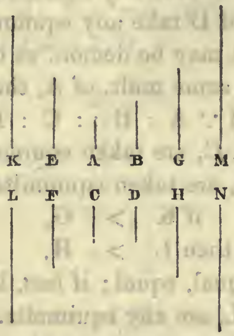
∵ EF is same mult. of A, that GH is of C,
∴ (No. mags. in EF which = A) = (No. mags. in GH which = C).
Divide EF into mags. EK, KF, ea. = A;
and GH into mags. GL, LH, ea. = C:
∴ No. mags. EK, KF = No. mags. GL, LH.
And ∵ A is same mult. of B, that C is of D,
and that EK = A,
and GL = C,
∴ EK is same mult. of B, that GL is of D:
similarly, KF is same mult. of B, that LH is of D.
And so on, if there are more parts in EF, GH which = A, C.
Now ∵ EK, 1st, is same mult. of B, 2d, that GL, 3d, is of D, 4th,
and that KF, 5th, is same mult. of B, 2d, that LH, 6th, is of D, 4th,
∴ EF, 1st + 5th, is same mult. of B, 2d, that GH, 3d + 6th, is of D, 4th. 2. 5.

If therefore, the first be the same multiple, &c. &c. Q. E. D.

PROP. IV.—Theorem.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz. 'the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.'

Let A, 1st, : B, 2d, = C, 3d, : D, 4th. And of A and C let there be taken any equimults. E, F; and of B and D any equimults. G, H, then E : G : : F : H.



Of E, F take any equimults. K, L;
and of G, H take any equimults. M, N:
then, ∵ E is same mult. of A, that F is of C,
and, that K is same mult. of E, that L is of F,
∴ K is same mult. of A, that L is of C. 3. 5.
Similarly, M is same mult. of B, that N is of D.
And, ∵ A : B : : C : D, hyp.
and, that K is same mult. of A, that L is of C,
and, that M is same mult. of B, that N is of D,
if K > M,
then L > N,
if equal, equal; if less, less. 5 def. 5.
But K is same mult. of E, that L is of F,
also M is same mult. of G, that N is of H,
∴ E : G : : F : H. 5 def. 5.
Therefore, &c. &c. Q. E. D.

Cor.

PROP. IV. CONTINUED.

Cor. Likewise if the first has the same ratio to the second, which the third has to the fourth, then also any equimultiples of the first and third have the same ratio to the second and fourth: and in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth.

Let A, 1st, : B, 2d, : : C, 3d, : D, 4th; and of A and C let E and F be any equimults. whatever; then E : B : : F : D.

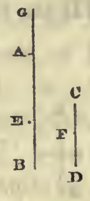
Of E and F take any equimults. K, L,
and of B and D take any equimults. G, H:
then it may be demon. as before,
that K is the same mult. of A, that L is of C:
and ∵ A : B : : C : D,
and, that of A, C, are taken equimults. K and L,
and of B, D, are taken equimults. G and H,
if K > G,
then L > H,
if equal, equal; if less, less. 5 def. 5.
Now K, L, are any equimults. of E, F,
and G, H, are any equimults. of B, D,
∴ E : B : : F : D.

And in the same way the other case may be demonstrated.

PROP. V.—Theorem.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder, that the whole is of the whole.

Let AB be the same mult. of CD that AE taken from 1st is of CF taken from 2d; then rem. EB is same mult. of rem. FD, that whl. AB is of whl. CD.



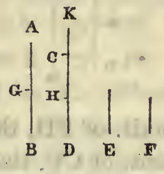
Take AG same mult. of FD, that AE is of CF,
∴ AE is same mult. of CF, that EG is of CD; 1. 5.
but, AE is same mult. of CF, that AB is of CD, hyp.
∴ EG is same mult. of CD, that AB is of CD;
∴ EG = AB; 1 ax. 5.
take away com. mag. AE,
then rem. AG = rem. EB;
and since AE is same mult. of CF, that AG is of FD,
and that AG = EB,
∴ AE is same mult. of CF, that EB is of FD:
but AE is same mult. of CF, that AB is of CD,
∴ EB is same mult. of FD, that AB is of CD.

Therefore, if any magnitudes, &c. &c. Q. E. D.

PROP. VI.—Theorem.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others, or equimultiples of them.

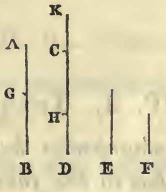
Let two mags. AB, CD be equimults. of two E, F, and AG, CH taken from the first two be equimults. of the same E, F. Then rems. GB, HD are either = E, F, or equimults. of them.



First.—Let GB = E.
Then HD = F.
Make CK = F.
And ∵ AG is same mult. of E, that CH is of F,
and that GB = E,
and CK = F,
∴ AB is same mult. of E, that KH is of F;
but AB is same mult. of E, that CD is of F,
∴ KH is same mult. of F, that CD is of F;
∴ KH = CD; 1 ax. 5.
take away com. mag. CH,
then rem. KC = rem. HD:
but KC = F, constr.
∴ HD = F.

Secondly,

PROP. VI.—CONTINUED.



SECONDLY.—Let GB be a mult. of E.

Then HD is same mult. of F, that GB is of E.

Make CK the same mult. of F, that GB is of E;

and ∵ AG is same mult. of E, that CH is of F,

and GB is same mult. of E, that CK is of F,

∴ AB is same mult. of E, that KH is of F: 2. 5.

but AB is same mult. of E, that CD is of F,

∴ KH is same mult. of F, that CD is of F;

∴ KH = CD; 1 ax. 5.

take from both, CH,

∴ rem. KC = rem. HD;

∴ HD is same mult. of F, that GB is of E.

Therefore if two magnitudes, &c. &c. Q. E. D.

PROP. A. Theorem.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then, if the first be greater than the second the third is also greater than the fourth; and if equal, equal; if less, less.

Take any equimults. of ea. of them, such as the doubles of ea.

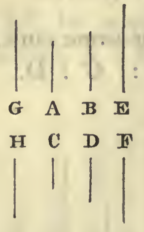
Then, if 2 first	$>$	2 second,
$\therefore$ 2 third	$>$	2 fourth :
but, if first	$>$	second,
then, 2 first	$>$	2 second ;
$\therefore$ also 2 third	$>$	2 fourth ;
and $\therefore$ third	$>$	fourth.
Similarly, if the first	$>$ or $<$	second,
then third	$>$ or $<$	fourth.

Therefore if the first, &c. &c. Q. E, D.

PROP. B.—Theorem.

If four magnitudes are proportionals, they are proportionals also when taken inversely.

If $A : B :: C : D$ then also inversely $B : A :: D : C$.



Of B and D take any equimults. E and F;
and of A and C any equimults. G and H.

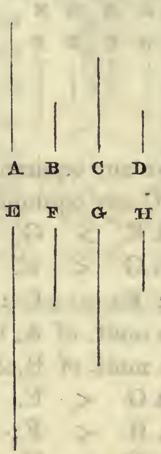
Let $E > G$,
then $G < E$.
And $\because A : B :: C : D$,
and that G is same mult. of A, 1st, that H is of C, 3rd,
and that E is same mult. of B, 2nd, that F is of D, 4th,
and, that $G < E$,
$\therefore H < F$; 5 def. 5.
i. e. $F > H$:
if, then $E > G$,
$\therefore F > H$.
Similarly if $E = G$,
then $F = H$,
and if less, less.
Now E is same mult. of B, that F is of D,
and G is same mult. of A. that H is of C,
$\therefore B : A :: D : C$.

Therefore if four magnitudes, &c. &c. Q, E. D.

PROP. C.—Theorem.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second, as the third is to the fourth.

First.—Let A, 1st, be same mult. of B, 2d, that C, 3d, is of D, 4th; then A : B :: C : D.



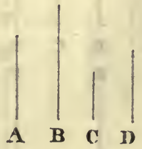
Of A and C, take any equimults. E and G;
and of B and D, take any equimults. F and H.
Then, $\because$ A is same mult. of B, that C is of D,
and, that E is same mult. of A, that G is of C,
$\therefore$ E is same mult. of B, that G is of D; 3. 5.
$\therefore$ E and G are the same mults. of B and D;
but F and H are equimults. of B and D:
then, if E be a mult. of B $>$ F is of B.
$\therefore$ G is a mult. of D $>$ H is of D,
i. e. if E $>$ F,
then G $>$ H.

Similarly,

PROP. C.—CONTINUED.

Similarly, if E = F,
then G = H,
and if less, less.
But, E and G are any equimults. of A and C,
and F and H are any equimults. of B and D,
∴ A : B :: C : D. 5 def. 5.

SECONDLY—Let A, 1st, be same part of B, 2nd, that C, 3d, is of D, 4th; also then A : B :: C : D.



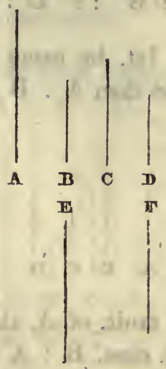
For, B is same mult. of A, that D is of C,
∴, by preced. case, B : A :: D : C,
and inversely A : B :: C : D. B. 5.

Therefore if the first, &c. &c. Q. E. D.

PROP. D.—Theorem.

If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.

Let A : B : : C : D; and First, let A be a mult. of B; then C is same mult. of D.



Take E = A;
and make F same mult. of D, that A or E is of B.
Then ∵ A : B : : C : D,
and that E and F are any equimults. of B, 2d, and D, 4th,
∴ A : E : : C : F; cor. 4. 5.
but A = E,
∴ C = F; A. 5.
and F is same mult. of D, that A is of B,
∴ C is same mult. of D, that A is of B.

Secondly—Let A be a part of B; then C is same part of D.

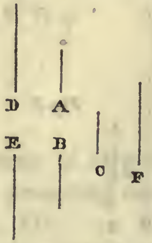
For, ∵ A : B : : C : D,
then, inversely, B : A : : D : C.
But A is a part of B,
∴ B is a mult. of A :
and by preced. case, D is same mult. of C, that B is of A,
i. e. C is same part of D, that A is of B.

Therefore, if the first, &c. &c. Q. E. D.

PROP. VII.—THEOREM.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Let A and B be equal mags., and C any other; then A : C :: B : C also C : A :: C : B.



FIRST—Of A and B take any equimults. D and E, and of C take any equimult. F.

Then, $\because$ D is same mult. of A, that E is of B,

and that $A = B$,

$\therefore D = E$; 1 ax. 5.

and, if $D > F$,

then $E > F$,

if equal, equal; and if less, less.

Now D and E are any equimults. of A and B,

and F is any mult. of C,

$\therefore A : C :: B : C$. 5 def. 5

SECONDLY—Also $C : A :: C : B$,

For with the same constr. it may be demon.

that $D = E$,

and $\therefore$ if $F > D$,

then $F > E$,

if equal, equal; if less, less.

Now F is any mult. of C,

and D and E any equimults. of A and B,

$\therefore C : A :: C : B$. 5 def. 5.

Therefore, equal magnitudes, &c. &c. Q. E. D.

PROP. VIII.—Theorem.

Of unequal magnitudes the greater has a greater ratio to the same than the less has: and the same magnitude has a greater ratio to the less than it has to the greater.

Let AB, BC be unequal mags. of which AB is the greater; and let D be any mag. whatever; then AB : D > BC : D, also D : BC > D : AB.

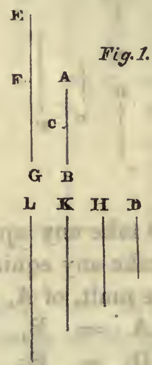


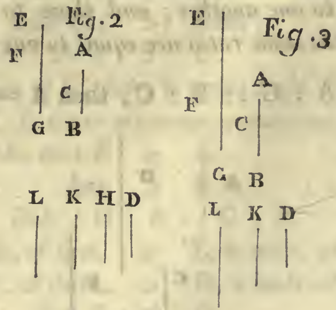
Fig. 1.

First—If that mag. which is $\ngtr$ other, of AC, CB, be $\nless$ D,
take EF, and FG = 2 AC, and 2 CB: fig. 1st,
but, if that which is $\ngtr$ other, of AC, CB be < D,
(as in figs. 2d and 3d),
then this mag. AC or CB can be multiplied
so as to become > D;
let it be mult. until it become > D;
and let the other be mult. as often.
And let EF be the mult. thus taken, of AC;
and FG the same mult. of CB:
∴ EF or FG > D.
Now in every one of the cases
take H = 2 D,
and K = 3 D,
and so on until the mult. of D be the first which becomes > FG:
let L be that mult. of D which is first > FG;

and

PROP. VIII. CONTINUED.

and K be the mult. of D which is next $<$ L.



Then $\because$ L is that mult. of D which first becomes $>$ FG,

$\therefore$ K, the next preceding mult. of D, is $\not>$ FG;

i. e. FG $\not<$ K.

And since EF is same mult. of AC, that FG is of CB,

$\therefore$ FG is same mult. of CB that EG is of AB; 1. 5.

$\therefore$ EG and FG are equimults. of AB and CB.

Now FG $\not<$ K, demon.

and EF $>$ D, constr.

$\therefore$ whl. EG $>$ K + D;

but K + D $=$ L,

$\therefore$ EG $>$ L;

but FG $\not>$ L,

and EG, FG are equimults. of AB and BC,

and L is a mult. of D,

$\therefore$ AB : D $>$ BC : D. 7 def. 5.

SECONDLY—D : BC $>$ D : AB.

For with same construction it may be demon.

that L $>$ FG;

but that L $\not>$ EG;

now L is a mult. of D,

and FG, EG are equimults. of CB, AB,

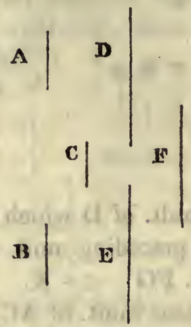
$\therefore$ D : BC $>$ D : AB.

Therefore, if unequal magnitudes, &c. &c. Q. E. D.

PROP. IX.—Theorem.

Magnitudes which have the same ratio to the same magnitudes are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

First—Let A : C :: B : C; then A = B.



For if A ≠ B,
one is > other;
let A > B.
Then there are some equimults. of A and B, 8. 5.
and some mult. of C,
such, that the mult. of A > the mult. of C;
but mult. of B ≯ mult. of C.
Let such mults. be taken:
and let D, E be equimults. of A, B;
and F a mult. of C;
so that D > F,
and E ≯ F.
But, ∵ A : C :: B : C,
and that D, E are equimults. of A, B,
and F is mult. of C,
and that D > F;
then also E > F; 5 def. 5.
but E ≯ F,
which is impossible.
∴ A is not ≠ B,
i. e. A = B.

Secondly,

PROP. IX. CONTINUED.

SECONDLY—Let C : A :: C : B; then also A = B.

For if A $\neq$ B,

then one > other;

let A > B.

Then of C, there is some mult. F,

and of A, B there are some equimults. D, E, 8. 5.

such, that F > E,

but $\ngtr$ D.

But $\because$ C : A :: C : B,

and that F, a mult. of first, > E, a mult. of second,

$\therefore$ F, a mult. of third, > D, a mult. of fourth; 5 def. 5.

But F $\ngtr$ D:

which is impossible.

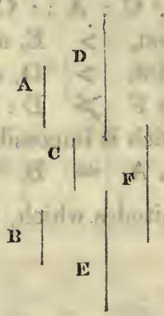
$\therefore$ A = B.

Wherefore magnitudes which, &c. &c. Q. E. D.

PROP. X.—Theorem.

That magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two: and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.

First—Let A : C > B : C; then A > B.



For, ∵ A : C > B : C,
∴ of A and B there are some equimults. D and E,
and of C some mult. F, 7 def. 5.
such, that D > F,
but E ≯ F;
and ∴ D > E:
and ∵ D, E are equimults. of A, B,
and that D > E,
∴ A > B. 4 ax. 5.

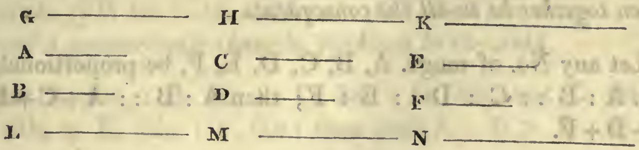
Secondly—Let C : B > C : A; then B < A.
For of C there is some mult. F,
and of B, A, some equimults. E, D, 7 def. 5.
such that F > E,
but ≯ D,
∴ E < D:
and ∵ E and D are equimults. of B and A,
∴ B < A. 4 ax. 5.

Therefore that magnitude, &c. &c. Q. E. D.

PROP. XI.—Theorem.

Ratios that are the same to the same ratio are the same to each other.

Let A : B :: C : D, and also C : D :: E : F; then shall A : B : E : F.



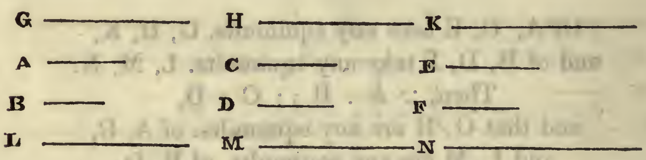
Of A, C, E take any equimults. G, H, K,
and of B, D, F take any equimults. L, M, N.
Then, ∵ A : B :: C : D,
and that G, H are any equimults. of A, C,
and L, M are any equimults. of B, D,
if G > L,
then H > M,
if equal, equal; if less, less. 5 def. 5.
Again, ∵ C : D :: E : F,
and that H, K are any equimults. of C, E,
and M, N are any equimults. of D, F,
if H > M,
then K > N,
and if equal, equal; if less, less. 5 def. 5.
But it has been shewn
that, if G > L,
then H > M,
if equal, equal; if less, less.
∴ if G > L,
K > N,
if equal, equal; if less, less.
Now G, K are any equimults. of A, E,
and L, N are any equimults. of B, F,
∴ A : B :: E : F. 5 def. 5.

Therefore ratios, &c. &c. Q. E. D.

PROP. XII.—Theorem.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let any No. of mags. A, B, C, D, E, F, be proportionals; i. e. $A : B :: C : D :: E : F$; then $A : B :: A+C+E : B+D+F$.



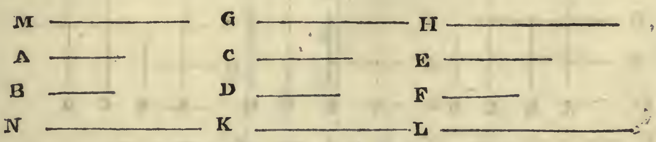
Of A, C, E take any equimults. G, H, K,
and of B, D, F take any equimults. L, M, N.
Then, $\because A : B :: C : D :: E : F$,
and that, G, H, K are equimults. of A, C, E,
and L, M, N are equimults. of B, D, F,
if $G > L$,
then $H > M$,
and $K > N$,
if equal, equal; if less, less. 5 def. 5.
$\therefore$ if $G > L$,
then $G+H+K > L+M+N$,
and if equal, equal; if less, less.
Now G and $G+H+K$ are any equim. of A and $A+C+E$, I. 5.
also L and $L+M+N$ are any equimults. of B and $B+D+F$,
$\therefore A : B :: A+C+E : B+D+F$.

Wherefore if any number, &c. &c. Q. E. D.

PROP. XIII.—THEOREM.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

Let A, 1st, : B, 2d, : : C, 3d, : D, 4th, but C, 3d, : D, 4th, $>$ E, 5th, : F, 6th; then shall A : B $>$ E : F.



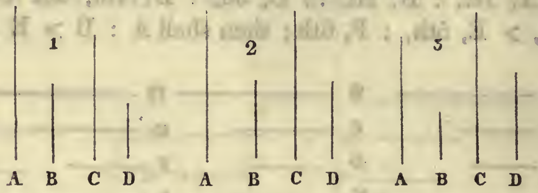
$\because$ C : D $>$ E : F,
there are some equimults. as G and H, of C and E,
and some equimults. as K and L, of D and F,
such, that G $>$ K,
but H $\ngtr$ L: 7 def. 5.
and take M, same mult. of A that G is of C;
and N, same mult. of B that K is of D.
Then, $\because$ A : B : : C : D,
and that M, G are equimults. of A, C,
and N, K are equimults. of B, D,
if M $>$ N,
then G $>$ K,
and if equal, equal; if less, less. 5 def. 5.
but G $>$ K, constr.
$\therefore$ M $>$ N;
but H $\ngtr$ L.
Now, M, H are equimults. of A, E,
and N, L are equimults. of B, F,
$\therefore$ A : B $>$ E : F. 7 def. 5.

Wherefore if the first, &c. &c. Q. E. D.

PROP. XIV.—Theorem.

If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let A, 1st, : B, 2d, : : C, 3rd, : D, 4th.



First—Let A > C; then B > D.

$\because$ A > C,

and B is another mag.

$\therefore$ A : B > C : B; 8. 5.

but A : B : : C : D,

$\therefore$ C : D > C : B; 13. 5.

$\therefore$ D < B; 10. 5.

i. e. B > D.

Secondly—Let A = C; then B = D.

For A : B : : C, i. e. A : D.

$\therefore$ B = D. 9. 5.

Thirdly—Let A < C; then B < D.

For, C > A;

and, $\because$ C : D : : A : B,

$\therefore$ D > B; 1st case.

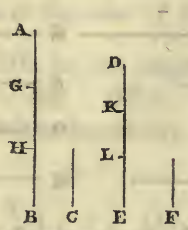
i.e. B < D.

Therefore, if the first, &c. &c. Q. E. D.

PROP. XV.—THEOREM.

Magnitudes have the same ratio to each other which their equimultiples have.

Let AB be the same mult. of C, that DE is of F; then C : F : : AB : DE.



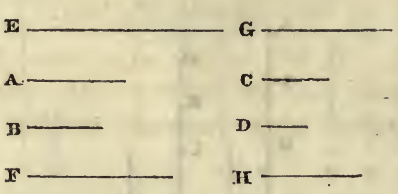
∵ AB is same mult of C that DE is of F,
∴ (No.mags.inABwhich=C) = (No.mags.inDE which=F).
Divide AB into mags. AG, GH, HB, ea. = C;
and DE into mags. DK, KL, LE, ea. = F;
∴ No. mags. AG, GH, HB = No. of mags. DK, KL, LE.
And ∵ AG, GH, HB = ea. other,
and that DK, KL, LE = ea. other,
∴ AG : DK : : GH : KL : : HB : LE; 7.5.
and ∴ AG : DK : : AB : DE. 12.5.
But AG = C,
and DK = F,
∴ C : F : : AB : DE.

Therefore magnitudes, &c. &c. Q. E. D.

PROP. XVI.—Theorem.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

Let A, B, C, D, be four proportionals; viz. A : B :: C : D, they are proportionals when taken alternately, i. e. A : C :: B : D.



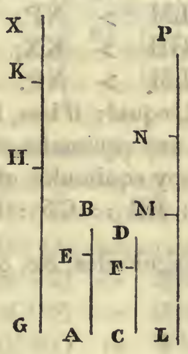
Of A, B take any equimults. E, F,
and of C, D take any equimults. G, H :
and ∵ E is same mult. of A, that F is of B,
∴ A : B :: E : F; 15. 5.
but A : B :: C : D,
∴ C : D :: E : F. 11. 5.
Again, ∵ G is same mult. of C, that H is of D,
∴ C : D :: G : H;
but C : D :: E : F,
∴ E : F :: G : H;
∴ if E > G,
then F > H,
if equal, equal; if less, less. 14. 5.
Now E, F are any equimults. of A, B,
and G, H, are any equimults. of C, D,
∴ A : C :: B : D. 5 def. 5.

If therefore four magnitudes, &c. &c. Q. E. D.

PROP. XVII.—Theorem.

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately: that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

Let AB, BE, CD, DF, be the mags. taken jointly, which are proportionals, i. e. AB : BE : : CD : DF; they shall also be proportionals taken separately, viz. AE : EB : : CF : FD.



Of AE, EB, CF, FD take any equimults. GH, HK, LM, MN; and again of EB, FD take any equimults. KX, NP.

And ∵ GH is same mult. of AE, that HK is of EB,
∴ GH is same mult. of AE, that GK is of AB; 1. 5.
but GH is same mult. of AE, that LM is of CF,
∴ GK is same mult. of AB, that LM is of CF.
Again, ∵ LM is same mult. of CF, that MN is of FD,
∴ LM is same mult. of CF, that LN is of CD; 1. 5.
but LM is same mult. of CF, that GK is of AB, demon.
∴ GK is same mult. of AB, that LN is of CD;
i. e. GK, LN are equimults. of AB, CD.

Next,

PROP. XVII.—CONTINUED.

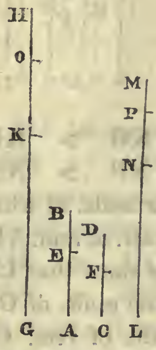
Next, ∵ HK is same mult. of EB, that MN is of FD,
and that KX is same mult. of EB, that NP is of FD,
∴ HX is same mult. of EB, that MP is of FD; 2. 5.
and ∵ AB : BE :: CD : DF,
and that GK, LN are equimults. of AB, CD,
and HX, MP are equimults. of EB, FD,
if GK > HX,
then LN > MP,
if equal, equal; if less, less. 5 def. 5.
But, if GH > KX,
add to both HK,
then GK > HX;
∴ also LN > MP;
take from both MN,
then LM > NP;
∴ if GH > KX,
then LM > NP,
if equal, equal; if less, less. 5 def. 5.
Now GH, LM are any equimults. of AE and CF,
and KX, NP are any equimults. of EB and FD,
∴ AE : EB :: CF : FD.

Therefore, if magnitudes, &c. &c. Q. E. D.

PROP. XVIII.—Theorem.

If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly: that is, if the first be to the second, as the third to the fourth, the first and second together shall be to the second, as the third and fourth together to the fourth.

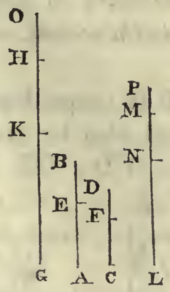
Let AE, EB, CF, FD be proportionals; that is, AE : EB :: CF : FD; they shall also be proportionals when taken jointly, viz. AB : BE :: CD : DF.



Of AB, BE, CD, DF take any equimults. GH, HK, LM, MN;
and again of BE, DF take any equimults. KO, NP.
And $\because$ KO, NP are equimults. of BE, DF,
and that KH, NM are also equimults. of BE, DF,
if KO, a mult. of BE, $>$ KH, also mult. of BE,
then NP, mult. of DF, $>$ NM, also mult. of DF;
and if KO $=$ KH,
then NP $=$ NM;
and if less, less. 5 def. 5.
First—Let KO $\not>$ KH;
$\therefore$ NP $\not>$ NM:
and $\because$ GH, HK are equimults. of AB, BE,
and that AB $>$ BE,
$\therefore$ GH $>$ HK; 3 ax. 5.
but KO $\not>$ KH,
$\therefore$ GH $>$ KO.
Similarly LM $>$ NP:
$\therefore$, if KO $\not>$ KH,

PROP. XVIII. CONTINUED.

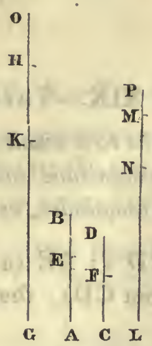
then GH, a mult. of AB, > KO, a mult. of BE.
Similarly LM, a mult. of CD, > NP, a mult. of DF.



SECONDLY—Let KO > KH;
∴ also NP > NM. demon.
And ∵ whl. GH is same mult. of whl. AB, that HK is of BE,
∴ rem. GK is same mult. of rem. AE, that GH is of AB; 5. 5.
which is the same that LM is of CD;
similarly, ∵ LM is same mult. of CD, that MN is of DF,
∴ rem. LN is same mult. of rem. CF, that whl. LM is of whl. CD. 5. 5.
But LM is same mult. of CD, that GK is of AE, demon.
∴ GK is same mult. of AE, that LN is of CF;
i. e. GK, LN are equimults. of AE, CF:
and ∵ KO, NP are equimults. of BE, DF,
and that KH, NM are also equimults. of BE, DF,
if KH, NM be taken from KO, NP,
∴ rems. HO, NP are either =, or equimults. of BE, DF. 6. 5.
First—Let HO, MP = BE, DF;
and, ∵ AE : EB :: CF : FD,
and that GK, LN are equimults. of AE, CF,
∴ GK : EB :: LN : FD: cor. 4. 5.
but HO = EB,
and MP = FD,
∴ GK : HO :: LN : MP:
if, ∴ GK > HO,
then LN > MP;
if equal, equal; if less, less. A. 5.

Secondly,

PROP. XVIII. CONTINUED.



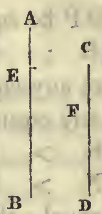
Secondly—Let HO, MP be equimults. of EB, FD:
and ∵ AE : EB : : CF : FD,
and that GK, LN are any equimults. of AE, CF,
and HO, MP are any equimults. of EB, FD;
if GK > HO,
then LN > MP;
if equal, equal; if less, less; 5 def. 5.
which was also shewn in preceding case:
if ∴ GH > KO,
take from both KH,
then GK > HO;
∴ also LN > MP;
and consequently, adding NM to both,
LM > NP:
if ∴ GH > KO,
then LM > NP;
similarly, if equal, equal; if less, less.
Now in the FIRST case,
where KO was assumed ≯ KH,
it was shewn that GH > KO always;
and also LM > NP;
but GH, LM are any equimults. of AB, CD,
and KO, NP are any equimults. of BE, DF,
∴ AB : BE : : CD : DF. 5 def. 5.

Therefore if magnitudes, &c. &c. Q. E. D.

PROP. XIX.—Theorem.

If a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the remainder, as the whole to the whole.

Let whl. AB : whl. CD : : AE (a mag. taken from AB) : CF, (a mag. taken from CD); then shall rem. EB : rem. FD : : AB : CD.



For, ∵ AB : CD : : AE : CF,		
∴ altern. AB : AE : : CD : CF;	16. 5.	
and divid. EB : FD : : AE : CF;	17. 5.	
again, altern. EB : AE : : FD : CF;		
but AE : CF : : AB : CD;	hyp.	
∴ EB : FD : : AB : CD.		

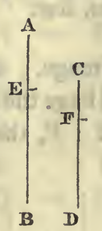
Therefore if the whole, &c. &c. Q. E. D.

Cor. If the whole be to the whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder likewise is to the remainder, as the magnitude taken from the first to that taken from the other. The demonstration is contained in the preceding.

PROP. E.—THEOREM.

If four magnitudes be proportionals, they are also proportionals by conversion, that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let AB : BE : : CD : DF ; then BA : AE : : DC : CF.



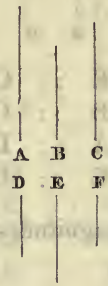
For, ∵	AB	:	BE	: :	CD : DF ;		
by div.	AE	:	EB	: :	CF : FD ;		17. 5.
by inver.	BE	:	EA	: :	DF : FC ;		13. 5.
∴ by compos.	BA	:	AE	: :	DC : CF.		18. 5.

Therefore, if four magnitudes, &c. &c. Q. E. D.

PROP. XX.—Theorem.

If there be three magnitudes, and other three, which, taken two and two, have the same ratio ; then, if the first be greater than the third, the fourth shall be greater than the sixth ; and if equal, equal ; and if less, less.

Let A, B, C, be three mags. ; and D, E, F, other three, which, taken two and two, have the same ratios, viz. A : B : : D : E ; and B : C : : E : F, then



First.—Let A > C ; then shall D > F.

∵ A > C,

and B any other mag.

∴ A : B > C : B ; 8. 5.

but D : E : : A : B,

∴ D : E > C : B ; 13. 5.

and ∵ B : C : : E : F,

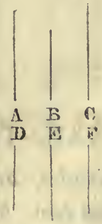
invert. C : B : : F : E,

∴ D : E > F : E ; cor. 13. 5.

∴ D > F. 10. 5.

Secondly,

PROP. XX. CONTINUED.



SECONDLY.—Let $A = C$; then shall $D = F$.

$\because A = C$,

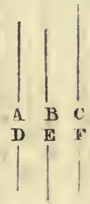
$\therefore A : B :: C : B$; 7. 5.

but $A : B :: D : E$,

and $C : B :: F : E$,

$\therefore D : E :: F : E$; 11. 5.

$\therefore D = F$. 9. 5.



THIRDLY.—Let $A < C$; then shall $D < F$.

For $C > A$;

and as by 1st, case $C : B :: F : E$,

similarly $B : A :: E : D$,

$\therefore$ by 1st, case $F > D$:

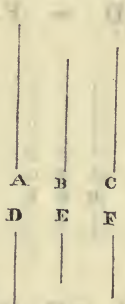
and $\therefore D < F$.

Therefore, if there be three magnitudes, &c. &c. Q. E. D.

PROP. XXI.—Theorem.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let A, B, C be three mags. and D, E, F three others, which have the same ratio, taken two and two, but in a cross order, viz. A : B : : E : F and B : C : : D : E; then



First—Let $A > C$; then shall $D > F$.

$\because A > C$,

and B is any other mag.

$\therefore A : B > C : B$; 8. 5.

but $E : F :: A : B$,

$\therefore E : F > C : B$: 13. 5.

and $\because B : C :: D : E$,

$\therefore$ invers. $C : B :: E : D$:

and $E : F > C : B$, demon.

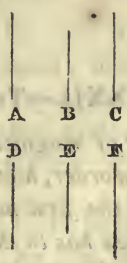
$\therefore E : F > E : D$; cor. 13.5.

and $\therefore F < D$; 10. 5.

i. e. $D > F$.

Secondly;

PROP. XXI. CONTINUED.



SECONDLY—Let $A = C$; then shall $D = F$.

$\because A = C,$

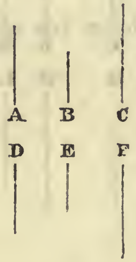
$\therefore A : B :: C : B;$ 7. 5.

but $A : B :: E : F,$

and $C : B :: E : D,$

$\therefore E : F :: E : D;$

$\therefore D = F.$ 9. 5.



THIRDLY—Let $A < C$; then shall $D < F$.

For $C > A,$

and, as was shewn, $C : B :: E : D;$

similarly $B : A :: F : E,$

$\therefore$, by 1st case, $F > D;$

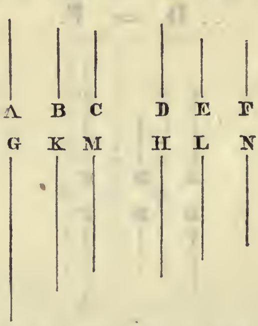
$\therefore D < F.$

Therefore if there be three magnitudes, &c. &c. Q. E. D.

PROP. XXII.—Theorem.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio: the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same. *N.B.* This is usually cited by the words "ex æquali," or, "ex æquo."

First.—Let there be three mags. A, B, C, and three others D, E, F, which, taken two and two, have the same ratio; i.e. A : B :: D : E, and B : C :: E : F. Then shall A : C :: D : F.



Of A and D take any equimults. G, H;
of B and E take any equimults. K, L;
and of C and F take any equimults. M, N.
Then, ∵ A : B :: D : E,
and that G, H are equimults. of A, D,
and K, L are equimults. of B, E.
∴ G : K :: H : L. 4. 5.
Similarly K : M :: L : N.

Now, ∵ there are three mags. G, K, M, and also three others H, L, N, which, taken two and two, have the same ratio;

if G > M,

then

PROP. XXII.—CONTINUED.

then H $>$ N;

if equal, equal; if less, less. 20. 5.

Now G, H are any equimults. of A, D,

and M, N, are any equimults. of C, F,

∴ A : C :: D : F. 5 def. 5.

SECONDLY.—Let A, B, C, D, be four mags. and four others E, F, G, H, which, taken two and two, have the same ratio; viz. A : B :: E : F; B : C :: F : G; and C : D :: G : H. Then shall A : D :: E : H.

For, ∵ A, B, C, are three mags. and E, F, G, three others which, taken two and two, have the same ratio,

∴, by 1st case, A : C :: E : G;

but C : D :: G : H;

∴ again, by 1st case A : D :: E : H.

A. B. C. D.
E. F. G. H.

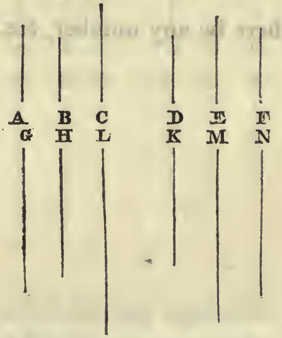
and so on, whatever be the number of mags.

Wherefore, if there be any number, &c. &c. Q. E. D.

PROP. XXIII.—Theorem.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same. *N.B.* This is usually cited by the words "ex æquali in proportione perturbatâ;" or "ex æquo perturbato."

First—Let there be three mags. A, B, C, and three others D, E, F, which taken two and two in cross order have same ratio, i. e. A : B : : E : F, and B : C : : D : E. Then A : C : : D : F.



Of A, B, D, take any equimults. G, H, K;
and of C, E, F, take any equimults. L, M, N.
And ∵ G, H, are equimults. of A, B,
∴ A : B : : G : H: 15. 5.
similarly E : F : : M : N:
but, A : B : : E : F,
∴ G : H : : M : N; 11. 5.
and ∵ B : C : : D : E,
and that H, K, are equimults. of B, D,

and

PROP. XXIII.—CONTINUED.

and L, M, are equimults. of C, E,

∴ H : L :: K : M: 4.5.

and it was shewn,

that G : H :: M : N.

Now ∵ there are three mags. G, H, L, and three others, K, M, N, which, taken two in cross order, have the same ratio;

if G > L,

then K > N;

if equal, equal; if less, less. 21.1.

Now G, K, are equimults. of A, D,

and L, N, are equimults. of C, F,

∴ A : C :: D : F.

SECONDLY—Let there be four mags. A, B, C, D, and four others E, F, G, H, which taken two and two in cross order, have the same ratio, viz. A : B :: G : H; B : C :: F : G, and C : D :: E : F. Then shall A : D :: E : H.

For, ∵ A, B, C are three mags. and F, G, H, are three others, which taken two and two in cross order, have the same ratio;

∴ by 1st case, A : C :: F : H;

but C : D :: E : F,

∴ by 1st case A : D :: E : H.

A. B. C. D.
E. F. G. H.

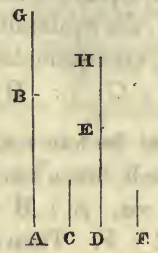
And so on, whatever be the number of mags.

Therefore, if there be any number, &c. &c. Q. E. D.

PROP. XXIV.—Theorem.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same which the sixth has to the fourth; the first and fifth together shall have to the second, the same ratio which the third and sixth together have to the fourth.

Let AB, 1st, : C, 2d, : : DE, 3rd, : F, 4th, and let BG, 5th, : C, 2d, : : EH, 6th, : F, 4th; then AG, 1st, + 5th, : C, 2d, : : DH, 3d, + 6th, : F, 4th.



∵ BG : C : : EH : F,
∴ invert. C : BG : : F : EH:
and ∵ AB : C : : DE : F,
and that C : BG : : F : EH,
∴ ex æquali, AB : BG : : DE : EH; 22. 5.
∴ compon. AG : GB : : DH : HE: 18. 5.
but GB : C : : HE : F,
∴ ex æquali AG : C : : DH : F.

Therefore, if the first, &c. &c. Q. E. D.

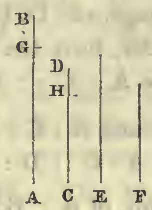
Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second, as the excess of the third and sixth to the fourth. The demonstration of this is the same with that of the proposition, if division be used instead of composition.

Cor. 2. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude; as is manifest.

PROP. XXV.—Theorem.

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.

Let the four mags. AB, CD, E, F, be proportionals, viz. AB : CD : : E : F; and let AB be greatest of them, and consequently F the least.* Then shall $AB + F > CD + E$. *A, 14 & 15. 5.



Take $AG = E$;
and $CH = F$.
Then, $\because AB : CD :: E : F$,
and that $AG = E$,
and $CH = F$,
$\therefore AB : CD :: AG : CH$;
and $\because$ whl. AB : whl. CD :: AG : CH,
$\therefore$ rem. GB : rem. HD :: whl. AB : whl. CD; 19. 5.
but $AB > CD$,
$\therefore GB > HD$; A. 5.
and $\because AG = E$,
and $CH = F$,
$\therefore AG + F = CH + E$.
If $\therefore$ $\left.\begin{matrix} AG + F, \\ CH + E, \end{matrix}\right\}$ be added to the unequal mags. $\left\{\begin{matrix} GB, \\ HD, \end{matrix}\right.$
then, $\because GB > HD$,
$\therefore AB + F > CD + E$.

Therefore, if four mags. &c. &c. Q. E. D.

PROP. F.—THEOREM.

Ratios which are compounded of the same ratios, are the same with each other.

Let A : B : : D : E, and B : C : : E : F; then the ratio which is comp. of A : B and B : C is the same with that which is comp. of D : E and E : F, i. e. A : C : : D : F.*

A.	B.	C.
D.	E.	F.

* def. of comp. ratio.

∵ A, B, C, are three mags. and D, E, F, three others, which, taken two and two in order, have the same ratio;

∴ ex æquo A : C : : D : F, 22. 5.

Next let A : B : : E : F, and B : C : D : E,
∴ ex æquo in pertur. A : C : : D : : F; 23. 5.

A.	B.	C.
D.	E.	F.

i. e. A : C, which is comp. of A : B, and B : C is the same with D : F,

which is comp. of D : E and E : F.

Q. E. D.

The proposition may be demonstrated similarly whatever be the number of ratios in either case.

PROP. G.—Theorem.

If several ratios be the same with several ratios, each to each; the ratio which is compounded of ratios which are the same with first ratios, each to each, is the same with the ratio compounded of ratios which are the same with the other ratios, each to each.

Let A : B : : E : F; and C : D : : G : H: and let A : B : : K : L; and C : D : : L : M; then shall K : M be comp.* of K : L and L : M which are the same with A : B and C : D. Also let E : F : : N : O; and G : H : : O : P; then shall N : P be comp. of N : O and O : P, which are the same with E : F and G : H. Now it is to be shewn that K : M is the same with N : P or that K : M : : N : P.

* def. of comp. ratio.

A. B. C. D.	K. L. M.
E. F. G. H.	N. O. P.

∵ K : L : : (A : B i.e. E : F i.e. : :) N : O,
and L : M : : (C : D and G : H and : :) O : P,
∴ ex æquali K : M : : N : P. 22. 5.

Therefore if several ratios, &c. &c. Q. E. D.

PROP. H. THEOREM.

If a ratio compounded of several ratios be the same with a ratio compounded of any other ratios, and if one of the first ratios, or a ratio compounded of any of the first, be the same with one of the last ratios, or with the ratio compounded of any of the last; then the ratio compounded of the remaining ratios of the first, or the remaining ratio of the first, if but one remain, is the same with the ratio compounded of those remaining of the last, or with the remaining ratio of the last.

Let the first ratios be those of A ; B, B : C, C : D, D : E and E : F; and let the others be those of G : H, H : K, K : L and L : M. Also let A : F (which is comp. of the first ratios*) be the same with G : M (which is the comp. of the other ratios). And also let A : D (which is comp. of A : B, B : C, C : D) be the same with G : K (which is comp. of G : H and H : K). Then shall the ratio comp. of the rem. first ratios, viz. D : F be the same with K : M, which is comp. of the rem. other ratios; i. e. D : F : : K : M.

A. B. C. D. E. F.
G. H. K. L. M.

* def. of comp. ratio.

∵ A : D	: :	G : K,		hyp.
∴ invers. D : A	: :	K : G :		B. 5.
And A : F	: :	G : M,		
∴ ex æquo. D : F	: :	K : M.		22. 5.

Therefore if a ratio, &c. &c. Q. E. D.

PROP. K.—Theorem.

If there be any number of ratios, and any number of other ratios such, that the ratio which is compounded of ratios which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios which are the same, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same to several of the first ratios, each to each, be the same to one of the last ratios, or to the ratio which is compounded of ratios which are the same, each to each, to several of the last ratios; then the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are the same, each to each, to the remaining ratios of the first, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio which is compounded of ratios which are the same, each to each, to these remaining ratios.

h, k, l.	
A, B; C, D; E, F.	S, T, V, X.
G, H; K, L; M, N; O, P; Q, R.	Y, Z, a, b, c, d.
e, f, g. m, n, o, p.	

Let A : B, C : D, E : F be the first ratios; and G : H, K : L, M : N, O : P, Q : R be the other ratios; and let A : B, : : S : T; and C : D : : T : V, and E : F : : V : X.

Therefore (by def. A. 5.) S : X is comp. of S : T, T : V, and V : X, which are the same with A : B, C : D, E : F, ea. to ea.

Also let G : H : : Y : Z; and K : L : : Z : a; M : N : : a : b;
O : P : : b : c; and Q : R : : c : d;

Therefore again (by same def.) Y : d is comp. of Y : Z, Z : a, a : b, b : c, and c : d, which are the same ea. to ea. with G : H, K : L, M : N, O : P, and Q : R;

∴ By hyp. S : X : : Y : d.

Also let A : B, i. e. S : T, which is one of the first ratios,

PROP. K. CONTINUED.

be the same with $e : g$, which is comp. of $e : f$ and $f : g$, which by hyp. are same with G : H, K : L, two of the other ratios;

And let the other $h : l$ be that which is compounded of $h : k$, $k : l$, which are the same with remaining first ratios, viz. C : D, and E : F;

Also let $m : p$ be that which is comp. of $m : n$, $n : o$, and $o : p$, which are the same ea. to ea. with the remaining other ratios, viz. M : N, O : P, Q : R;

Then $h : l :: m : p$

	h, k, l.	
	A, B; C, D; E, F.	S, T, V, X.
G, H; K, L; M, N; O, P; Q, R.		Y, Z, a, b, c, d.
e, f, g.	m, n, o, p.	

$\because e : f ::$ (G : H i.e. ::) Y : Z,

and $f : g ::$ (K : L i.e. ::) Z : a,

$\therefore$ ex æquali $e : g ::$ Y : a:

And A : B i.e. S : T :: $e : g$, hyp.

$\therefore$ S : T :: Y : a;

and invers. T : S :: a : Y;

and S : X :: Y : d,

$\therefore$ ex æquali T : X :: $a : d$.

Also $\because h : k ::$ (C : D i.e. ::) T : V,

and $k : l ::$ (E : F i.e. ::) V : X,

$\therefore$ ex æquali $h : l ::$ T : X.

In the same manner it may be demon.

that $m : p :: a : d$,

And it was shewn that T : X :: $a : d$,

$\therefore h : l :: m : p$. 11.5.

Q. E. D.

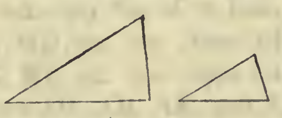
The propositions G, K, are usually, for the sake of brevity, expressed in the same terms with propositions F and H: and therefore it was proper to shew the true meaning of them when they are so expressed; especially since they are very frequently made use of by geometers.

BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.



II.

" Reciprocal figures, viz. triangles and parallelograms, are
" such as have their sides about two of their angles propor-
" tionals in such a manner, that a side of the first figure
" is to the side of the other, as the remaining side of this
" other is to the remaining side of the first.*

---

* The definition of reciprocal figures appears to be useless. Dr. Simpson is inclined to think it not genuine, and gives, in his note on the place, another definition, which, with a trifling alteration, is the following:

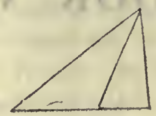
" Two magnitudes are said to be reciprocally proportional to two others, when one of the first is to one of the others, as the remaining one of the last is to the remaining one of the first."

III.

A right line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

IV.

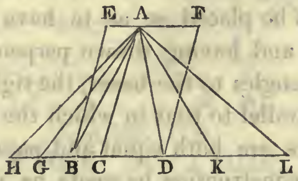
The altitude of any figure is the right line drawn from its vertex perpendicular to the base.



PROP. I.—Theorem.

Triangles and parallelograms of the same altitude are to each other as their bases.

Let the △s ABC, ACD, and the ▱s EC, CF have the same altit. viz. the ⊥ drawn from A to BD; then the base BC : base CD : : △ ABC : △ ACD : : ▱ EC : ▱ CF.



Prod. BD both ways to pts. H, L;
take any No. of rt. lines,
viz. { BG, GH, ea. = base BC,
and DK, KL, ea. = base CD;
join AG, AH, AK, and AL.
Then, ∵ CB, BG, GH = ea. other,
∴ △s AHG, AGB, ABC = ea. other 38. 1.
∴ △ AHC is same mult. of △ ABC that base HC is of base BC;
similarly, △ ALC is same mult. of △ ADC that base LC is of base DC:
and if HC = CL,
then △ AHC = △ ALC, 38. 1.
and if greater, greater; if less, less.
Now ∵, of BC and △ ABC, 1st and 3d, are taken any equimults. HC, and △ AHC,
and of CD and △ ACD, 2d and 4th, are taken any equimults. CL and △ ALC,
and that, if HC > CL,
then △ AHC > △ ALC,
if equal, equal; if less, less;

∴ base

PROP. I.—CONTINUED.

∴ base BC : base CD	: :	△ ABC : ACD.		5 def. 5.
And ∵ □ CE	=	2 △ ABC,	}	41. 1.
and that □ CF	=	2 △ ACD,	}	
∴ △ ABC : △ ACD	: :	□ CE : □ CF :		15. 5.
and ∴ also, BC : CD	: :	△ ABC : △ ACD,		
∴ base BC : base CD	: :	□ CE : □ CF.		11. 5.

Therefore, triangles, &c. &c. Q. E. D.

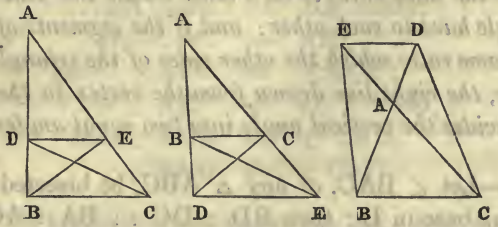
Cor. From this, it is plain, that triangles and parallelograms which have equal altitudes, are to each other as their bases.

Let the figures be placed so as to have their bases in the same right line; and having drawn perpendiculars from the vertices of the triangles to the bases, the right line which joins the vertices is parallel to that in which the bases are, because the perpendiculars are both equal and parallel to each other: then if the same construction be made as in the proposition, the demonstration will be the same.

PROP. II.—Theorem.

If a right line be drawn parallel to one of the sides of triangle, it shall cut the other sides, or these produced, proportionally: and if the sides, or the sides produced, be cut proportionally, the right line which joins the points of section shall be parallel to the remaining side of the triangle.

First—Let DE be drawn ∥ BC, a side of △ ABC; then BD : DA : : CE : EA.



Join BE, CD.

Then △ BDE = △ CDE, 37. 1.

(for they are on same base DE and between same ∥ s DE, BC,)

and ∴ △ ADE is another mag.

∴ △ BDE : △ ADE : : △ CDE : △ ADE; 7. 5.

but △ BDE : △ ADE : : BD : DA, 1. 6.

(for they have same alt. DE).

Similarly △ CDE : △ ADE : : CE : EA,

∴ BD : DA : : CE : EA. 11. 5.

Secondly—Let AB, AC sides of △ ABC, or these prod. be cut in pts. D and E, so that BD : DA : : CE : EA; then DE ∥ BC.

The same construc. being made.

∵ BD : DA : : CE : EA.

and BD : DA : : △ BDE : △ ADE, } 1. 6.

and that CE : EA : : △ CDE : △ ADE, }

∴ △ BDE : △ ADE : : △ CDE : △ ADE;

i. e. △ s BDE, CDE have same ratio △ ADE;

and ∴ △ BDE = △ CDE; 9. 5.

and they are on same side of base DE;

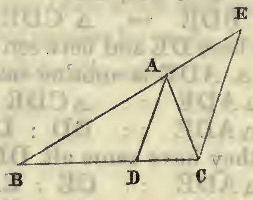
∴ DE ∥ BC. 39. 1.

Wherefore if a right line, &c. &c. Q. E, D.

PROP. III.—Theorem.

If the angle of a triangle be divided into two equal angles, by a right line which also cuts the base, the segments of the base shall have the same ratio to each other which the other sides of the triangle have to each other: and if the segments of the base have the same ratio which the other sides of the triangle have to each other, the right line drawn from the vertex to the point of section, divides the vertical angle into two equal angles.

First—Let ∠ BAC, of any △ ABC, be bisected by AD, cutting the base in D; then BD : DC : : BA : AC.



Thro. C, draw CE ∥ DA; 31. 1.
and let BA prod. meet CE in E;
and ∵ AC falls on ∥ s AD, EC,
∴ ∠ ACE = ∠ CAD; 29. 1.
but ∠ CAD = ∠ BAD, hyp.
∴ ∠ BAD = ∠ ACE.
Again ∵ BE falls on ∥ s AD, DE,
∴ ex. ∠ BAD = int. ∠ AEC;
but ∠ BAD = ∠ ACE,
∴ ∠ ACE = ∠ AEC;
and ∴ side AE = side AC. 6. 1.
And ∵ AD ∥ EC a side of △ BCF,
∴ BD : DC : : BA : AE; 2. 6.
but AE = AC,
∴ BD : DC : : BA : AC. 7. 5

Secondly—

PROP. III.—CONTINUED.

SECONDLY—Let BD : DC : : BA : AC; join AD; then ∠ BAC is bis. by AD, i. e. ∠ BAD = ∠ CAD.

The same constr. being made,

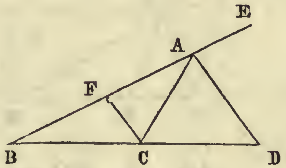
∵ BD : DC	: :	BA : AC,	
and that BD : DC	: :	BA : AE,	2. 6.
(for AD	∥	EC,)	
∴ BA : AC	: :	BA : AE;	11. 5.
and ∴ AC	=	AE;	9. 5.
and ∴ ∠ AEC	=	∠ ACE;	5. 1.
but ∠ AEC	=	ex. ∠ BAD,	29. 1.
also ∠ ACE	=	∠ CAD,	29. 1.
∴ ∠ BAD	=	∠ CAD.	

Wherefore, if the angle, &c. &c. Q. E. D.

PROP. A.—Theorem.

If the outward angle of a triangle made by producing one of its sides, be divided into two equal angles, by a right line which also cuts the base produced; the segments between the dividing line and the extremities of the base have the same ratio which the other sides of the triangle have to each other: and if the segments of the base produced have the same ratio which the other sides of the triangle have, the right line drawn from the vertex to the point of section divides the outward angle of the triangle into two equal angles.

First—Let ex. ∠ CAE of any △ ABC be bis. by AD which meets the base produced in D; then BD : DC : : BA : AC.



Thro. C, draw CF	∥	AD.	31. 1.
And ∵ AC falls on		∥ s AD, FC,	
∴ ∠ ACF	=	∠ CAD:	29. 1.
but ∠ CAD	=	∠ DAE,	hyp.
∴ ∠ DAE	=	∠ ACF;	
and ∵ FE falls on		∥ s AD, FC,	
∴ ex. ∠ DAE	=	int. ∠ CFA:	
but ∠ ACF	=	DAE,	
∴ ∠ ACF	=	∠ CFA;	
∴ AF	=	AC:	6. 1.
and ∵ AD	∥	FC a side of △ BCF,	
∴ BD : DC	: :	BA : AF:	2. 6.
now AF	=	AC,	
∴ BD : DC	: :	BA : AC.	

Secondly,

PROP. A. CONTINUED.

SECONDLY—Let BD : DC : : BA : AC; then ∠ EAD = CAD.

The same construct. being made,

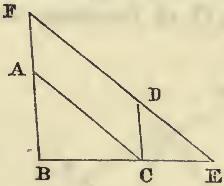
∵ BD : DC	: :	BA : AC,	
and that BD : DC	: :	BA : AF,	11.5.
∴ BA : AC	: :	BA : AF;	
∴ AC	=	AF;	9.5.
∴ ∠ AFC	=	∠ ACF:	5.1.
but ∠ AFC	=	ex. ∠ EAD,	29.1.
also ∠ ACF	=	alt. ∠ CAD,	29.1.
∴ ∠ EAD	=	∠ CAD.	

Wherefore the outward angle, &c. &c. Q. E. D.

PROP. IV.—THEOREM.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides, that is, are the antecedents or consequents of the ratios.

Let ABC, DCE be equiang. $\triangle$s, having $\angle$ ABC = $\angle$ DCE and $\angle$ ACB = $\angle$ DEC and consequently $\angle$ BAC* = $\angle$ CDE. Then the sides about the equal * 32. 1. $\angle$s of $\triangle$s ABC, DCE are proportionals; and those are the homologous sides which are opposite to the equal $\angle$s.



Let $\triangle$ DCE be so placed, that its side CE may be contiguous to and in the same rt. line with BC.

$\because \angle$ ABC + $\angle$ ACB < 2 rt. $\angle$s, 17. 1.

and that $\angle$ ACB = $\angle$ DEC,

$\therefore \angle$ ABC + $\angle$ DEC < 2 rt. $\angle$s;

and $\therefore$ BA, ED, if produced far enough, will meet. 12 ax. 1.

Let BA, ED be prod. to meet in F:

and $\because \angle$ ABC = $\angle$ DCE, hyp.

$\therefore$ BF $\parallel$ CD. 28. 1.

Again, $\because \angle$ ACB = $\angle$ DEC,

$\therefore$ AC $\parallel$ FE; 28. 1.

$\therefore$ fig. FC is a $\square$;

and $\therefore$ AF = CD; } 34. 1.

and AC = FD: }

and $\because$ AC $\parallel$ FE a side of $\triangle$ FBE,

$\therefore$ BA : AF :: BC : CE; 2. 6.

but

PROP. IV. CONTINUED.

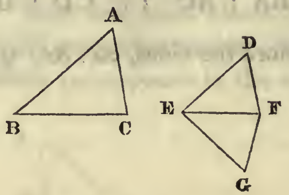
but AF			=	CD,			
∴ BA	:	CD	: :	BC	:	CE;	7. 5.
and altern. AB	:	BC	: :	DC	:	CE.	
Again, ∵ CD			‖	BF,			
∴ BC	:	CE	: :	FD	:	DE;	2. 6.
but FD			=	AC,			
∴ BC	:	CE	: :	AC	:	DE;	
and altern. BC	:	CA	: :	CE	:	ED.	
Now ∵ AB	:	BC	: :	DC	:	CE,	demon.
and that BC	:	CA	: :	CE	:	ED,	
∴ also ex æquali. BA	:	AC	: :	CD	:	DE.	22. 5.

Therefore the sides, &c. &c. Q. E. D.

PROP. V.—Theorem.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have their equal angles opposite to the homologous sides.

Let △s ABC, DEF have their sides proportionals, so that AB : BC : : DE : EF; and BC : CA : : EF : FD; and consequently ex æquali BA : AC : : ED : DF. Then △ ABC is equiang. to △ DEF, and their equal ∠s are opp. to the homologous sides, viz. ∠ ABC = ∠ DEF, ∠ BCA = ∠ EFD also ∠ BAC = ∠ EDF.



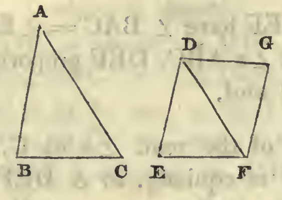
At E and F, in EF, make { ∠ FEG	=	∠ ABC,	} 23. 1.
and ∠ EFG	=	∠ BCA;	}
then rem. ∠ EGF	=	rem. ∠ BAC;	32. 1.
and ∴ △ ABC is equiang. to △ GEF;			
and ∴ AB : BC	: :	GE : EF:	4. 6.
but AB : BC	: :	DE : EF,	hyp.
∴ DE : EF	: :	GE : EF;	11. 5.
∴ DE	=	GE;	9. 5.
similarly DF	=	FG:	
and ∵ DE	=	EG,	
and EF is com. to △s DEF, GEF,			
and base DF	=	base FG,	
∴ ∠ DEF	=	∠ GEF;	8. 1.
and consequently ∠ DFE	=	∠ GFE; }	4. 1.
and ∠ EDF	=	∠ EGF: }	
and ∵ ∠ DEF	=	∠ GEF,	
and that ∠ GEF	=	∠ ABC,	
∴ ∠ ABC	=	∠ DEF.	
similarly { ∠ ACB	=	∠ DFE,	
and ∠ BAC	=	∠ EDF,	
∴ △ ABC is equiang. to △ DEF.			

Wherefore if the sides, &c. &c. Q. E. D.

PROP. VI.—Theorem.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Let the △s ABC, DEF have the ∠ BAC in one = ∠ EDF in the other, and the sides about those ∠s proportionals; i. e. BA : AC : : DE : DF. Then △s ABC, DEF are equiang. and have ∠ ABC = ∠ DEF and ∠ ACB = ∠ DFE.



At D and F in DF, make { ∠ FDG	=	∠ BAC or EDF, }	23. 1.
{ and ∠ DFG	=	∠ ACB; }	
∴ rem. ∠ at B	=	rem. ∠ at G;	32. 1.
and ∴ △ ABC is equiang. to △ DGF;			
and ∴ BA : AC	: :	GD : DF :	4. 6.
but BA : AC	: :	ED : DF,	
∴ ED : DF	: :	GD : DF;	11. 5.
∴ ED	=	GD :	9. 5.
and ∵ DF is com. to △s EDF, GDF,			
then ED, DF	=	GD, DF ea. to ea. :	
and ∵ ∠ EDF	=	∠ GDF,	constr.
∴ base EF	=	base FG;	
and △ EDF	=	△ GDF, }	
and ∴ also ∠ DFG	=	∠ DFE, }	4. 1.
and ∠ DGF	=	∠ DEF; }	
but ∠ DFG	=	∠ ACB,	
∴ ∠ ACB	=	∠ DFE;	
also ∠ BAC	=	∠ EDF.	hyp.
∴ rem. ∠ ABC	=	rem. ∠ DEF;	
and ∴ △ ABC is equiang. to △ DEF.			

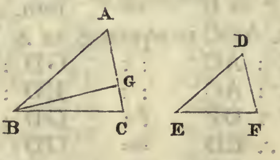
Wherefore if two triangles, &c. &c. Q. E. D.

PROP. VII.—Theorem.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; then, if each of the remaining angles be either less, or not less, than a right angle, or if one of them be a right angle; the triangles shall be equiangular, and shall have those angles equal about which the sides are proportionals.

Let $\triangle$s ABC, DEF have $\angle$ BAC $=$ $\angle$ EDF, and the sides about the two other $\angle$s ABC, DEF proportionals, i. e. AB : BC : : DE : EF; and,

First—Let ea. of the rem. $\angle$s at C, F be $<$ rt. $\angle$. Then the $\triangle$ ABC is equiang. to $\triangle$ DEF, viz. $\angle$ ABC $=$ $\angle$ DEF, and rem. $\angle$ at C $=$ rem. $\angle$ at F.



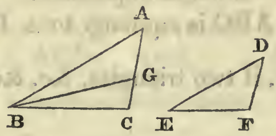
For if $\angle$ ABC $\neq$ $\angle$ DEF,
then one $>$ other;
let $\angle$ ABC $>$ $\angle$ DEF.
At B, in AB, make $\angle$ ABG $=$ $\angle$ DEF; 23. 1.
and $\because$ $\angle$ BAC $=$ $\angle$ EDF,
and that $\angle$ ABG $=$ $\angle$ DEF,
$\therefore$ rem. $\angle$ AGB $=$ rem. $\angle$ DFE; 32. 1.
$\therefore$ $\triangle$ ABG is equiang. to $\triangle$ DEF;
$\therefore$ AB : BG : : DE : EF; 4. 6.

but

PROP. VII. CONTINUED.

but $AB : BC :: DE : EF$,
$\therefore AB : BC :: AB : BG$; 11. 5.
$\therefore BC = BG$; 9. 5.
and $\therefore \angle BGC = \angle BCG$; 5. 1.
but $\angle BCG <$ rt. $\angle$, hyp.
$\therefore$ also $\angle BGC <$ rt. $\angle$;
and $\therefore$ adjac. $\angle BGA >$ rt. $\angle$; 13. 1.
but $\angle AGB = \angle DFE$, demon.
$\therefore \angle DFE >$ rt. $\angle$;
but $\angle DFE <$ rt. $\angle$, hyp.
which is absurd.
$\therefore \angle ABC$ is not $\neq \angle DEF$,
i. e. $\angle ABC = \angle DEF$;
and $\angle$ at $A = \angle$ at D;
$\therefore$ rem. $\angle$ at $C =$ rem. $\angle$ at F;
$\therefore \triangle ABC$ is equiang. to $\triangle DEF$.

SECONDLY—Let ea. of the $\angle$s at C, F be $\nless$ rt. $\angle$; then $\triangle ABC$ is equiang. to $\triangle DEF$.

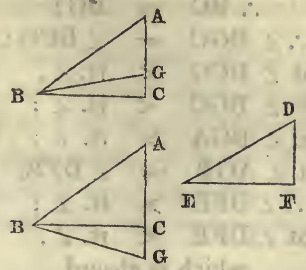


The same constr. being made it may be proved as before,
that $BC = BG$,
and $\therefore \angle BCG = \angle BGC$; 5. 1.
but $\angle BCG \nless$ rt. $\angle$,
$\therefore \angle BGC \nless$ rt. $\angle$;
$\therefore$ in $\triangle BGC$ are two $\angle$s BCG, BGC together $\nless$ 2 rt. $\angle$s;
which is impossible.
And $\therefore$ it may be proved as in 1st case,
that $\triangle ABC$ is equiang. to $\triangle DEF$.

THIRDLY,

PROP. VII. CONTINUED.

THIRDLY—Let one of the ∠s at C, F, viz. ∠ at C, be a rt. ∠ : then likewise △ ABC is equiang. to △ DEF.



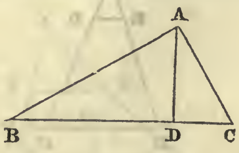
For if △ ABC is not equiang. to △ DEF;
then at B in AB, make ∠ ABG = ∠ DEF :
and it may be proved as in 1st case,
that, BG = BC,
and ∴ ∠ BCG = ∠ BGC; 5. 1.
but ∠ BCG is a rt. ∠,
∴ ∠ BGC is a rt. ∠ ;
∴ in △ BGC are two ∠s, BCG + BGC ≮ 2 rt. ∠s;
which is impossible. 17. 1.
∴ △ ABC is equiang. to △ DEF.

Wherefore, if two triangles, &c. &c. Q. E. D.

PROP. VIII.—THEOREM.

In a right angled triangle, if a perpendicular be drawn from the right angle to the base; the triangles on each side of it are similar to the whole triangle, and to each other.

Let ABC be a rt. ∠d △, having the rt. ∠ BAC, and from pt. A, let AD be drawn ⊥ base BC; then △s ABD, ADC, are simil. to the whl. △ ABC, and to each other.



∵ ∠ BAC = ∠ ADB, 11 ax. 1.
and that ∠ ABC is com. to △s ABC, ABD,
∴ rem. ∠ ACB = rem. ∠ BAD; 32. 1.
∴ △ ABC is equiang. to △ ABD;
and their sides about the = ∠s are proportional, 4. 6.
∴ △ ABC simil. △ ABD: 1 def. 6.
similarly △ ADC is equiang. and simil. △ ABC;
now ∵ △ ABD, or ADC is equiang. and simil. △ ABC,
∴ △ ABD simil. △ ADC.

Therefore, in a right angled triangle, &c. &c. Q. E. D.

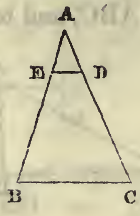
Cor. From this it is manifest that the perpendicular, drawn from the rt. ∠, of a rt. ∠d △, to the base, is a mean proportional between the segments of the base; and also that ea. of the sides is a mean proportional between the base, and its segment adjacent to that side;

Because, in △s BDA, ADC.—BD : DA : : DA : DC;
and in the △s ABC, DBA.—BC : BA : : BA : BD;
and in the △s ABC, ACD.—BC : CA : : CA : CD.

PROP. IX.—Problem.

From a given right line to cut off any part required.

Let AB be the given rt. line; it is required to cut off any part from it.



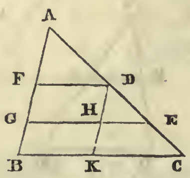
From pt. A, draw AC, making any ∠ with AB;
in AC take any pt. D;
and take AC, same mult. of AD, that AB is of part to be cut off;
join BC;
draw DE ‖ BC; 31.1.
then AE is the part required to be cut off.
∵ ED ‖ BC a side of △ ABC,
∴ CD : DA : : BE : EA; 2.6.
but compon. CA : AD : : BA : AE, 18.5.
∴ BA is same mult. of AE that CA is of AD; D.5.
and ∴ AE is same part of BA that AD is of CA.

Therefore, from AB, the part required is cut off. Q. E. F.

PROP. X.—Problem.

To divide a given right line similarly to a given divided right line, that is, into parts that shall have the same ratios to each other which the parts of the divided given right line have.

Let AB be the right line given to be divided, and AC the divided rt. line ; it is required to divide AB similarly to AC.



Let AC be divided in pts. D, E ;
and let AB, AC be placed so as to contain any ∠ ;
join BC ;

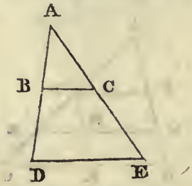
thro. D, E draw DF, EG	‖	BC ; }	31. 1.
and thro. D draw DHK	‖	AB ; }	
∴ ea. fig. FH, HB	is a	▱ ;	
∴ DH	=	FG, }	34. 1.
and HK	=	GB : }	
and ∵ HE	‖	KC a side of △ DKC,	
∴ CE : ED	: :	KH : HD ;	2. 6.
but KH	=	BG,	
and HD	=	GF,	
∴ CE : ED	: :	BG : GF.	
Again ∵ FD	‖	EG a side of △ AGE,	
∴ ED : DA	: :	GF : FA ;	
also CE : ED	: :	BG : GF.	demon.

Therefore, AB is divided similarly to AC. Q. E. F.

PROP. XI.—Problem.

To find a third proportional to two given right lines.

Let AB, AC be the two given rt. lines, and let them be placed so as to contain any $\angle$; it is required to find a third proportional to AB, AC.



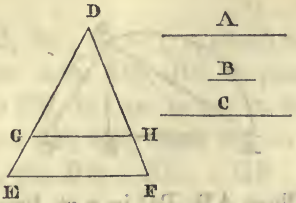
Prod. AB, AC to pts. D, E;
make BD $=$ AC; 3. 1.
join BC;
thro. D draw DE $\parallel$ BC. 31. 1.
Then, $\because$ BC $\parallel$ DE a side of $\triangle$ ADE,
$\therefore$ AB : BD : : AC : CE; 2. 6.
but BD $=$ AC,
$\therefore$ AB : AC : : AC : CE.

Therefore to AB, AC a third proportional CE is found. Q. E. F.

PROP. XII.—Problem.

To find a fourth proportional to three given right lines.

Let A, B, C be the three given rt. lines; it is required to find a fourth proportional to them.



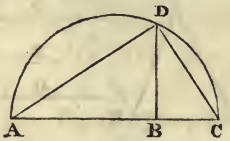
Take two rt. lines, DE, DF containing any ∠ EDF;
in these make DG = A,
GE = B,
and DH = C:
join GH;
thro. E draw EF ∥ GH.
Then ∵ GH ∥ EF a side of △ DEF,
∴ DG : GE :: DH : HF; 2. 6.
but DG = A,
GE = B,
and DH = C,
∴ A : B :: C :: HF.

Therefore, to A, B, C a fourth proportional HF has been found. Q. E. F.

PROP. XIII.—Problem.

To find a mean proportional between two given right lines.

Let AB, BC be the two given rt. lines; it is required to find a mean proportional to them.



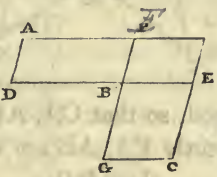
Place AB, BC in a rt. line;
On AC descr. ½ ⊙ ADC;
from B draw BD at rt. ∠ s to AC; 11. 1.
join AD, DC.
And ∵ ∠ ADC, in a ½ ⊙, is a rt. ∠, 31. 3.
and, that in rt. ∠ d △ ADC, DB is drawn from rt. ∠ ⊥ base,
∴ DB is a mean propor. between AB, BC segs. of base. cor. 8. 6.

Therefore between the given rt. lines AB, BC a mean proportional DB is found. Q. E. F.

PROP. XIV.—Theorem.

Equal parallelograms, which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to each other.

First—Let AB, BC, be = □ s, which have their ∠ s at B equal; and let the sides DB, BE be placed in the same rt. line, therefore also FB, BG are in one rt. line;* then the sides of the □ s AB, BC, about the = ∠ s, are reciprocally proportional, viz. DB : BE : : GB : BF. *14. 1.



Complete the □ FE.
And ∵ □ AB = □ BC, hyp.
and that EF is another mag.
∴ AB : FE : : BC : FE; 7. 5.
but AB : FE : : DB : BE, 1. 6.
also BC : FE : : GB : BF,
∴ DB : BE : : GB : BF, 11. 5.
∴ sides of □ s AB, BC, about = ∠ s, are reciprocally proportional.

Secondly—Let the sides about the equal ∠ s be reciprocally proportional, viz. DB : BE : : GB : BF; then □ AB = □ BC.

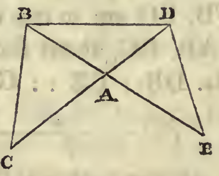
∵ DB : BE : : GB : BF,
and DB : BE : : □ AB : □ FE,
and GB : BF : : □ BC : □ FE
∴ AB : FE : : BC : FE; 11. 5.
∴ □ AB = □ BC. 9. 5.

Wherefore, equal parallelograms, &c. &c. Q. E. D.

PROP. XV.—Theorem.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional: and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to each other.

First—Let ABC, ADE be = $\triangle$s, which have $\angle$ BAC = $\angle$ DAE; then the sides about = $\angle$s are reciprocally proportional.—Viz. CA : AD : : EA : AB.



Let the $\triangle$s be placed, so that CA, AD be in one rt. line.

And consequently EA, AB are in one rt. line. 14. 1.

Join BD.

And, $\because$ $\triangle$ ABC = $\triangle$ ADE,

and that $\triangle$ ABD is another mag.

$\therefore$ CAB : BAD : : EAD : DAB; 7. 5.

but CAB : BAD : : base CA : base AD,
and EAD : DAB : : base EA : base AB, } 1. 6.

$\therefore$ CA : AD : : EA : AB. 11. 5.

$\therefore$ sides of the $\triangle$s, about = $\angle$s, are reciprocally propor.

Secondly—Let the sides of the $\triangle$ ABC, ADE, about the = $\angle$s, be reciprocally proportional, viz. CA : AD : : EA : AB; then $\triangle$ ABC = $\triangle$ ADE.

Join BD as before.

And $\because$ CA : AD : : EA : AB,

and that CA : AD : : $\triangle$ ABC : $\triangle$ BAD,
and EA : AB : : $\triangle$ EAD : $\triangle$ BAD, } 1. 6.

$\therefore$ ABC : BAD : : EAD : BAD; 11. 5.

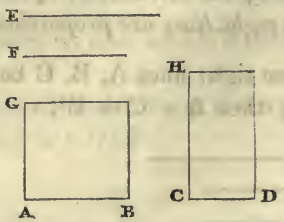
$\therefore$ $\triangle$ ABC = $\triangle$ AED. 9. 5.

Therefore equal triangles, &c. &c. Q. E. D.

PROP. XVI.—Theorem.

If four right lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four right lines are proportionals.

First—Let the four rt. lines AB, CD, E, F be proportionals, viz. AB : CD : : E : F. Then AB × F = CD × E.



From A, C draw AG, CH rt. ∠s to AB, CD;
make AG = F;
and CH = E;
and complete □s BG, DH.
And ∵ AB : CD : : E : F,
and that E = CH,
and F = AG,
∴ AB : CD : : CH : AG; 7. 5.
∴ sides of □s BG, DH, about = ∠s, are reciprocally propor.
∴ □ BG = □ DH; 14. 6.
but □ BG = AB × F,
(for AG = F,)
also □ DH = CD × E,
(for CH = E,)
∴ AB × F = CD × E.

Secondly—Let AB × F = CD × E; then AB : CD : : E : F.

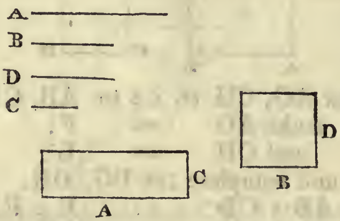
The same construc. being made;
∵ AB × F = CD × E,
and that □ BG = AB × F,
and □ DH = CD × E,
∴ □ BG = □ DH;
and they are equiangular;
∴ AB : CD : : CH : AG: 14. 6.
but CH = E,
and AG = F,
∴ AB : CD : : E : F.

Wherefore, if four right lines, &c. &c. Q. E. D.

PROP. XVII.—Theorem.

If three right lines be proportionals, the rectangle contained by the extremes is equal to the square of the mean: and if the rectangle contained by the extremes be equal to the square of the mean, the three right lines are proportionals.

First—Let three right lines A, B, C be proportionals, i.e. $A : B :: B : C$; then $A \times C = B^2$.



Take $D = B$;

then $A : B :: D : C$; 7. 5.

$\therefore A \times C = B \times D$: 16. 6.

but $B \times D = B^2$,

(for $D = B$,)

$\therefore A \times C = B^2$.

Secondly—Let $A \times C = B^2$; then $A : B :: B : C$.

make the same construction;

$\because A \times C = B^2$,

and that $B^2 = B \times D$,

(for $B = D$,)

$\therefore A \times C = B \times D$;

$\therefore A : B :: D : C$; 16. 6.

but $B = D$,

$\therefore A : B :: B : C$.

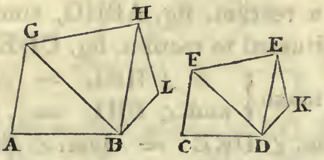
Wherefore if right lines, &c. &c. Q. E. D.

PROP. XVIII.—Problem.

On a given right line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure.

Let AB be the given rt. line, and CDEF the given rectilin. fig. of four sides;

First—It is required to descr. on AB a rectilin. fig. simil. and similarly situated to CDEF.



Join DF;

And at A, B, in AB make { ∠ BAG = ∠ FCD; and ∠ ABG = ∠ CDF; } 23. 1.

and ∴ rem. ∠ AGB = rem. ∠ CFD; 32. 1.

and ∴ △ FCD is equiang. to △ ABG.

Again, at G, B, in GB make { ∠ BGH = DFE, and ∠ GBH = FDE;

∴ rem. ∠ FED = rem. ∠ GHB;

and ∴ △ FDE is equiang. to △ GBH.

Then ∵ ∠ AGB = ∠ CFD,

and that also ∠ BGH = ∠ DFE,

∴ whl. ∠ AGH = whl. ∠ CFE:

similarly ∠ ABH = ∠ CDE;

also ∠ GAB = ∠ FCD;

and ∠ GHB = ∠ FED;

∴ rectilin. fig. ABHG is equiang. to rectilin. fig. CDEF.

And also these figs. have their sides about = ∠ s, propors.

For, ∵ △ GAB is equiang. to △ FCD,

∴ BA : AG :: DC : CF: 4. 6.

and

PROP. XVIII. CONTINUED.

and ∵ AG : GB : : CF : FD,
and that GB : GH : : FD : FE,
(for △ BGH is equiang. to △ DFE,)
∴ ex æquali. AG : GH : : CF : FE: 22. 5.
similarly { AB : BH : : CD : DE,
and GH : HB : : FE : ED. 4. 6.

Now, ∵ fig. ABHG is equiang. to the fig. CDEF,
and that both have their sides about = ∠s propors.
∴ rectilin. fig. ABHG simil. rectilin. fig. CDEF.

SECONDLY—It is required to descr. on AB a rectilin. fig. simil. given rectilin. fig. CDKEF of *five* sides.

Join DE;

On AB descr. a rectilin. fig. ABHG, simil. and similarly situated to rectilin. fig. CDEF; 1st case.

At B, H, in BH make { ∠ HBL = ∠ EDK;
and ∠ BHL = ∠ DEK;
∴ rem. ∠ DKE = rem. ∠ BLH. 32. 1.
And ∵ fig. ABHG simil. fig. CDEF,
∴ ∠ GHB = FED;
but also ∠ BHL = ∠ DEK, constr.
∴ whl. ∠ GHL = whl. ∠ FEK:
similarly ∠ ABL = ∠ CDK,
∴ rectilin. fig. AGHLB is equiang. to rectilin. fig. CFEKD.
And ∵ fig. ABHG simil. fig. CDEF,
∴ GH : HB : : FE : ED;
and HB : HL : : ED : EK, 4. 6.
∴ ex æquali. GH : HL : : FE : EK: 22. 5.
similarly { AB : BL : : CD : DK,
BL : LH : : DK : KE,
(for △ BLH is equiang. to △ DKE).
Now, ∵ rtlin. fig. AGHLB is equiang. to rtlin. fig. CFEKD,
and that they have their sides about = ∠s propors.,
∴ fig. AGHLB simil. fig. CFEKD.

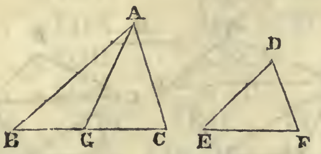
And in the same manner a rectilin. fig. may be descr. simil. and similarly situated to a given rectilin. fig. of six or more sides.

Q. E. F.

PROP. XIX.—Theorem.

Similar triangles are to each other in the duplicate ratio of their homologous sides.

Let ABC, DEF be similar △s, having ∠ B = ∠ E; and let AB : BC : : DE : EF, so that side BC is homol. to EF.* Then △ ABC : △ DEF : : dupl. of BC : EF. * 12 def. 5.



Take BG a third propor. to BC, EF, 11. 6.
so that BC : EF : : EF : BG;
Join GA.
Then ∵ AB : BC : : DE : EF,
∴ altern. AB : DE : : BC : EF; 16. 5.
but BC : EF : : EF : BG,
∴ AB : DE : : EF : BG; 11. 5.
∴ sides of △s ABG, DEF about =∠s are reciprocally propor.
∴ △ ABG = △ DEF; 15. 6.
and ∵ BC : EF : : EF : BG,
∴ BC : BG : : dupl. of BC : EF; 10 def. 5.
but BC : BG : : △ ABC : △ ABG, 1. 6.
∴ △ ABC : △ ABG : : dupl. of BC : EF;
but △ ABG = △ DEF,
∴ △ ABC : △ DEF : : dupl. of BC : EF.

Therefore similar triangles, &c. &c. Q. E. D.

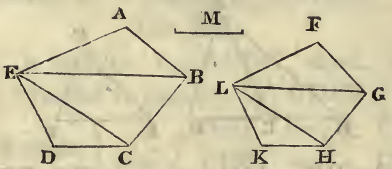
Cor. From this it is manifest, that if three right lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar and similarly described triangle upon the second.

PROP. XX.—Theorem.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to each other that the polygons have; and the polygons have to each other the duplicate ratio of that which their homologous sides have.

Let ABCDE, FGHKL be similar polygons, and let AB, FG be the homol. sides. Then

First—The polygons ABCDE, FGHKL may be divided into any No. of similar $\triangle$ s.



Join BE, EC; GL, LH;
and $\because$ fig. ABCDE simil. fig. FGHKL,
$\therefore \angle$ BAE = $\angle$ GFL; 1 def. 6.
and $\therefore$ BA : AE :: GF : FL; 1 def. 6.
and consequently $\triangle$ ABE is equiang. to $\triangle$ FGL; 6. 6.
and $\therefore$ also $\triangle$ ABE simil. $\triangle$ FGL; 4. 6.
$\therefore \angle$ ABE = $\angle$ FGL.
Again, $\because$ fig. ABCDE simil. fig. FGHKL,
$\therefore$ whl. $\angle$ ABC = whl. $\angle$ FGH; 1 def. 6.
and $\therefore$ rem. $\angle$ EBC = rem. $\angle$ LGH:
and $\because$ $\triangle$ ABE simil. $\triangle$ FGL,
$\therefore$ EB : BA :: LG : GF: 1 def. 6.
also $\because$ fig. ABCDE simil. fig FGHKL,
$\therefore$ AB : BC :: FG : GH; 1 def. 6.
$\therefore$ ex æquali EB : BC :: LG : GH; 22. 5.
i. e. sides about = $\angle$ s are proportionals;
$\therefore$ $\triangle$ EBC is equiang. to $\triangle$ LGH; 6. 6.
and conseq. $\triangle$ EBC simil. $\triangle$ LGH: 4. 6.
similarly $\triangle$ ECD simil. $\triangle$ LHK.
$\therefore$ The similar polygons ABCDE, FGHKL are $\div$ into same No. of similar $\triangle$ s.

Secondly

PROP. XX. CONTINUED.

SECONDLY—These $\triangle$s have ea. to ea. the same ratio which the polygons have to ea. other, the antecs. being $\triangle$s ABE, EBC, ECD, and conseqs. $\triangle$s FGL, LGH, LHK; also ABCDE : FGHKL : : dupl. of AB : FG.

$\because$ $\triangle$ABE simil. $\triangle$FGL,
$\therefore$ $\triangle$ABE : $\triangle$FGL : : dupl. of BE : GL : } 19. 6.
similarly $\triangle$EBC : $\triangle$LGH : : dupl. of BE : GL; }
$\therefore$ $\triangle$ABE : $\triangle$FGL : : $\triangle$EBC : $\triangle$LGH. 11. 5.
Again, $\because$ $\triangle$EBC simil. $\triangle$LGH,
$\therefore$ $\triangle$EBC : $\triangle$LGH : : dupl. of EC : LH,
Similarly $\triangle$ECD : $\triangle$LHK : : dupl. of EC : LH,
and $\therefore$ $\triangle$EBC : $\triangle$LGH : : $\triangle$ECD : $\triangle$LHK; 11. 5.
but $\triangle$EBC : $\triangle$LGH : : $\triangle$ABE : $\triangle$FGL; demon.
$\therefore$ABE : FGL : : EBC : LGH : : ECD : LHK;
$\therefore$ ABE : FGL : : fig. ABCDE : fig. FGHKL,
(for one antec. : its conseq. : : all antecs. : all conseqs.); 12. 5.
but $\triangle$ABE : : $\triangle$FGL : : dupl. of AB : FG,
and $\therefore$ ABCDE : FGHKL : : dupl. of AB : FG.

Wherefore similar polygons, &c. &c. Q. E. D.

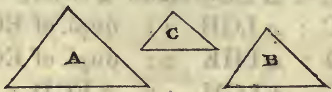
Cor. 1. In like manner it may be proved that similar four-sided figures, or of any number of sides, are to each other in the duplicate ratio of their homologous sides: and it has already been proved in triangles: therefore, universally, similar rectilineal figures are to each other in the duplicate ratio of their homologous sides.

Cor. 2. And if to AB, FG, two of the homologous sides, a third proportional M be taken, AB has to M the duplicate ratio of that which AB has to FG: but the four-sided figure or polygon upon AB, has to the four-sided figure or polygon upon FG, likewise, the duplicate ratio of that which AB has to FG; therefore, as AB is to M, so is the figure upon AB to the figure upon FG: which was also proved in triangles: therefore, universally, it is manifest, that if three right lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

PROP. XXI.—Theorem.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to each other.

Let ea. of rectilin. figs. A, B, be similar to rectilin. fig. C ; then fig. A similar fig. B.



∵ A simil. C,

∴ A is equiang. to C ;

and ∴ they have their sides about = ∠ s propors. 1 def. 6.

Again, ∵ B simil. C,

∴ B is equiang. to C ;

and ∴ they have their sides about = ∠ s propors. ;

∴ ea. of figs. A, B is equiang. to the fig. C,

and, of ea. of them and of C, the sides about = ∠ s are propors.

∴ fig. A is equiang. to fig. B, 1 ax. 1.

and have their sides about = ∠ s proportionals ; 11. 5.

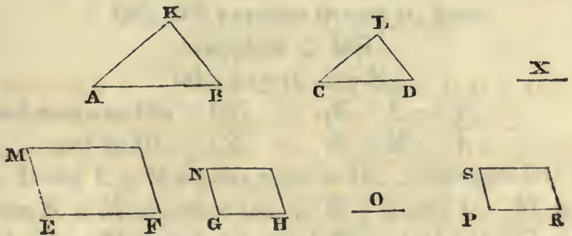
and ∴ rectilin. fig. A simil. rectilin. B. 1 def. 6.

Therefore, rectilineal figures, &c. &c. Q. E. D.

PROP. XXII.—Theorem.

If four right lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four right lines be proportionals, those right lines shall be proportionals.

First—Let the four rt. lines AB, CD, EF, GH be proportionals, i.e. AB : CD : : EF : GH, and on AB, CD let the similar rectilin. figs. KAB, LCD be similarly described; and on EF, GH, the similar rectilin. figs. MF, NH in like manner. Then rectilin. fig. KAB : LCD : : MF : NH.



To AB, CD take a third propor. X;
and to EF, GH take a third propor. O; } 11. 6.

and, ∵ AB : CD : : EF : GH,

and that CD : X : : GH : O, 11. 5.

∴ ex aquali. AB : X : : EF : O; 22. 5.

but AB : X : : KAB : LCD,
and EF : O : : MF : NH, } 2 cor. 20. 6.

∴ KAB : LCD : : MF : NH. 11. 5.

Secondly—Let rectilin. fig. KAF : LCD : : MF : NH, then shall AB : CD : : EF : GH.

Make AB : CD : : EF : PR; 12. 6.

and on PR descr. rectilin. fig. SR,

so that

PROP. XXII. CONTINUED.

so that SR be simil. and similarly situat. to MF, or NH. 18. 6.

Then, ∵ AB : CD :: EF : PR,

∴ by 1st case KAB : LCD :: MF : SR;

but KAB : LCD :: MF : NH, hyp.

∴ NH = SR: 9. 5.

and these are also simil. and similarly situated:

∴ GH = PR.

And ∵ AB : CD :: EF : PR,

and that PR = GH,

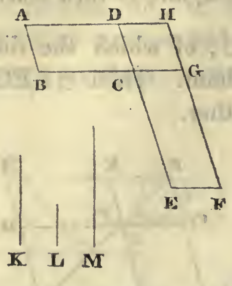
∴ AB : CD :: EF : GH.

Therefore, if four right lines, &c. &c. Q. E. D.

PROP. XXIII.—Theorem.

Equiangular parallelograms have to each other the ratio which is compounded of the ratios of their sides.

Let AC, CF be equiang. □s. having ∠ BCD = ∠ ECG. Then □ AC : □ CF is same with the ratio which is compounded of the ratio of their sides, i. e. BC : CE, which is the same with BC : CG and DC : CE * * def. A. 5.



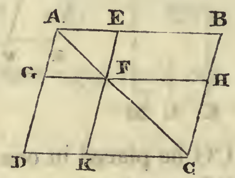
Let BC, CG be placed in one rt. line;
∴ DC, DE are also in one rt. line. 14. 1.
Complete □ DG,
take any rt. line K;
and make as BC : CG : : K : L;
and as DC : CE : : L : M; } 12. 6.
∴ K : L and L : M are the same as BC : CG and DC : CE :
now K : M is compound. of K : L and L : M, A. def. 5.
∴ also K : M is compound. of BC : CG and DC : CE :
and ∵ BC : CG : : □ AC : □ CH, 1. 6.
and that BC : CG : : K : L,
∴ K : L : : □ AC : □ CH. 11. 5.
Again, ∵ DC : CE : : □ CH : □ CF,
and that DC : CE : : L : M,
∴ L : M : : □ CH : □ CF; 11. 5.
and since also K : L : : □ AC : □ CH,
∴ ex æquali. K : M : : □ AC : □ CF : 22. 5.
but K : M is compounded of BC : CG and DC : CE,
consequently K : M : : BC : CE; A. def. 5.
∴ also □ AC : □ CF : : BC : CE;
i. e. □ AC : □ CF is same as the ratio which is compounded of the ratio of their sides.

Wherefore, equiangular parallelograms, &c. &c. Q. E. D.

PROP. XXIV.—Theorem.

Parallelograms about the diameter of any parallelogram, are similar to the whole, and to each other.

Let ABCD be a ▭, of which the diam. is AC; and EG, HK ▭s about the diam. Then ▭s EG, HK are similar ▭ ABCD, and to ea. other.



∵ DC	∥	GF,	
∴ ∠ ADC	=	∠ AGF:	29. 1.
similarly ∠ ABC	=	∠ AEF;	
and ea. of the ∠s BCD, EFG	=	opp. ∠ BAD;	34. 1.
∴ ∠s BCD, EFG	=	ea. other;	

and ∴ ▭s BD, EG are equiang.

and ∵ ∠ ABC	=	∠ AEF,	

and that ∠ BAC is com. to △s BAC, EAF,

∴ △s BAC, EAF are equiang.

∴ AB : BC	: :	AE : EF;	4. 6.
but BC	=	AD,	34. 1.
and EF	=	AG,	
∴ AB : AD	: :	AE : AG:	7. 5.
and similarly CD : DA	: :	FG : GA;	

∴ sides of ▭s BD, EG about = ∠s are proportionals;

and ∴ ▭s BD, EG simil. ea. other: 1 def. 6.

similarly ▭ BD simil. ▭ KH;

∴ ▭s EG, or KH simil. ▭ BD;

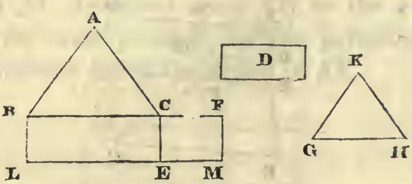
and ∴ ▭ EG simil. ▭ KH. 21. 6.

Wherefore the parallelograms, &c. &c. Q. E. D.

PROP. XXV.—Problem.

To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

Let ABC be given rectilin. fig., to which, the fig. to be described, is required to be similar, and D that, to which it must be equal; required to descr. a rectilin. fig. similar to ABC and = D.



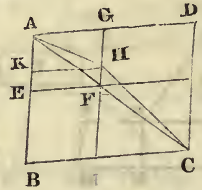
On BC descr. □ BE,
so that □ BE = fig. ABC; cor. 45. 1.
and on CE descr. □ CM,
so that □ CM = fig. D,
and having ∠ FCE = ∠ CBL;
∴ BC and CF are in one rt. line,
and also LE and EM. } 29. and 14. 1.
Between BC and CF find a mean propor. GH; 13. 6.
and on GH descr. rectilin. fig. KGH,
so that KGH be simil. and simil. situat. to rectilin. fig. ABC. } 18. 6.
Now, ∵ BC : GH :: GH : CF,
∴ BC : CF :: fig. ABC : KGH; 2 cor. 20. 6.
but BC : CF :: □ BE : □ EF, 1. 6.
∴ ABC : KGH :: BE : EF; 11. 5.
but ABC = BE, constr.
∴ KGH = EF; 14. 5.
but EF = D,
∴ KGH = D,
and also KGH simil. ABC.

Therefore a rectilin. fig. KGH is drawn simil. given rectilin. fig. ABC and = given rectilin. fig. D. Q. E. F.

PROP. XXVI.—Theorem.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Let the ▱s BD, EG be similar and similarly situated, and have ∠ DAB com.; then ▱s BD, EG are about same dia.



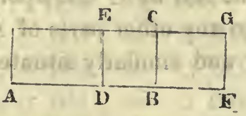
For, if not, if possible,
let ▱ BD have the dia. AHC,
but in a different direction from AF, the dia. of ▱ EG.
Let GH meet AHC in H;
thro. H draw HK ∥ AD or BC;
∴ ▱s BD, GK are about same dia. AHC;
and ∴ ▱s BD, GK simil. ea. other; 24. 6.
and ∴ DA : AB : : GA : AK : 1 def. 6.
and ∵ ▱s BD, EG simil. ea. other, hyp.
∴ DA : AB : : GA : AE;
and ∴ GA : AE : : GA : AK; 11. 5.
∴ AK = AE;
i. e. less = greater,
which is impossible.
∴ ▱s BD, GK are not about same dia.
and ∴ ▱s BD, EG must be about same dia.

Therefore, if two similar parallelograms, &c. &c. Q. E. D.

'To understand the three following propositions more 'easily, it is to be observed,

1. 'That a parallelogram is said to be applied to a right 'line, when it is described upon it as one of its sides. Ex. gr. 'the parallelogram AC is said to be applied to the right line 'AB.

2. 'But a parallelogram AE is said to be applied to a right 'line AB, deficient by a parallelogram, when AD the base of 'AE is less than AB, and therefore AE is less than the 'parallelogram AC described upon AB in the same angle, 'and between the same parallels, by the parallelogram DC; 'and DC is therefore called the defect of AE.

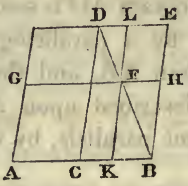


3. 'And a parallelogram AG is said to be applied to a right 'line AB, exceeding by a parallelogram, when AF the base 'of AG is greater than AB, and therefore AG exceeds AC 'the parallelogram described upon AB in the same angle, 'and between the same parallels, by the parallelogram BG.'

PROP. XXVII.—Theorem.

Of all parallelograms applied to the same right line and deficient by parallelograms, similar and similarly situated to that which is described upon the half of the line; that which is applied to the half and is similar to its defect, is the greatest.

Let AB be a rt. line bisected in C; and let □ AD be applied to the half, AC; which is therefore deficient from the □ upon the whl. line AB by □ CE upon the other half, CB. Of all □s applied to any other parts of AB, and deficient by □s that are similar and similarly situated to CE, AD is the greatest.



Let AF be any □ applied to AK, any other part of AB but its half, and so as to be deficient from □ AE by □ KH similar and similarly situated to □ CE : AD > AF.

First—Let AK, base of AF > AC, the ½ of AB.

And ∵ □ CE simil. □ KH,

∴ they are about the same dia. 26. 6.

draw dia. DB and complete the diagr.

And ∵ □ CF = □ FE, 43. 1.

add to ea. □ KH,

∴ whl. □ CH = whl. □ KE;

but □ CH = CG, 36. 1.

(for base AC = base CB,)

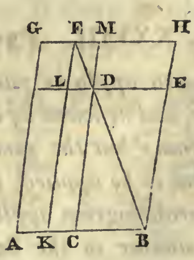
∴ □ CG = □ KE;

add to ea. □ CF,

∴ whl.

PROP. XXVII. CONTINUED.

∴ whl. □ AF = gnom. CHL;
∴ □ CE or □ AD > □ AF.



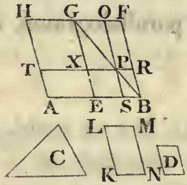
SECONDLY—Let AK < AC;
and ∵ BC = CA,
∴ HM = MG; 34. 1.
and ∴ □ DH = □ DG; 36. 1.
and ∴ □ DH > □ LG;
now □ DH = □ DK, 43. 1.
∴ □ DK > □ LG;
add to ea. the □ AL,
∴ whl. □ AD > whl. □ AF.

Therefore of all parallelograms, &c. &c. Q. E. D.

PROP. XXVIII.—Problem.

To a given right line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram similar to a given parallelogram: but the given rectilineal figure, to which the parallelogram to be applied, is to be equal, must not be greater than the parallelogram applied to half of the given line, having its defect similar to the defect of that which is to be applied: that is, to the given parallelogram.

Let AB be the given rt. line, and C the given rectilin. fig. which must not be > □ applied to $\frac{1}{2}$ of the given line, having its defect from that upon the whole line similar to the defect of that which is to be applied; and let D be the □ to which this defect is required to be similar. It is required to apply a □ to AB which shall = fig. C, and be deficient from the □ upon whl. line by a □ similar □ D.



Bis. AB in E; 10. 1.

on EB descr. □ EF,

so that EF be simil. and similarly situat. to □ D; 18. 6.

complete □ AG.

Now AG must be either = or > C;

and if AG = C,

then that is done which was required.

But, if □ AG ≠ C,

then □ AG > C:

and □ EF = □ AG, 36. 1.

∴ □ EF > C:

make

PROP. XXVIII. CONTINUED.

make □ KM = □ EF − C, 25. 6.
so that KM be simil. and similarly situat. to □ D;
but □ D simil. □ EF,
∴ □ KM simil. □ EF: 21. 6.
Let the side KL be homol. to EG,
and let LM be homol. to GF:
and ∵ □ EF = C + KM,
∴ □ EF > □ KM;
∴ EG > KL;
and ∴ GF > LM:
make GX = KL;
and GO = LM;
and complete □ XO;
∴ XO is = and simil. to KM;
but □ KM simil. □ EF,
∴ □ XO simil. □ EF;
∴ □s XO, EF are about same dia. 26. 6.
Let GPB be their dia. and complete the diagr.
Then, ∵ □ EF = C + KM,
and part XO = part KM,
∴ rem. gno. ERO = rem. fig. C:
and, ∵ □ OR = □ XS, 34. 1.
add to ea. □ SR,
∴ whl. □ OB = whl. □ XB;
but □ XB = □ TE, 36. 1.
(for base AE = base EB,)
∴ □ TE = □ OB;
add to ea. □ XS,
∴ whl. □ TS = whl. gnom. ERO;
but ERO = C,
∴ □ TS = C.

Therefore to the rt. line AB a □ TS is applied = given rectilin. fig. C and deficient by □ SR, simil. given □ D, ∵ SR simil. EF.*

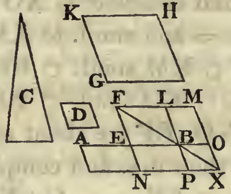
* 24. 6.

Q. E. F.

PROP. XXIX.—Problem.

To a given right line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram similar to another given.

Let AB be the given rt. line, and C the given rectilin. fig. to which, the □ to be applied, is required to be equal, and D the □, to which, the excess of the one to be applied above that upon AB, is required to be similar. It is required to apply to the given rt. line a □ = C, exceeding by a □ simil. D.



Bis. AB in E;
on EB descr. □ EL,
so that EL be simil. and similarly situat. to D;
make □ GH = □ EL + fig. C, 25.6
and also simil. and similarly situat. to D;
∴ □ GH simil. □ EL. 21.6.
Let the side KH be homol. to FL;
and KG be homol. to FE.
And ∵ □ GH > □ EL,
∴ KH > FL,
and KG > FE:
prod. FL and FE;
and make FLM = KH;
and FEN = KG
and complete □ MN;
∴ □ MN simil. □ GH;
but □ GH simil. □ EL,
∴ □ MN simil. □ EL:

and

PROP. XXIX. CONTINUED.

and ∴ EL and MN are about same dia. 26. 6.
draw their dia. FX and complete the diagr.
And since □ GH = □ EL+C,
and that GH = MN,
∴ MN = EL+C;
take away the com. □ EL,
∴ rem. gno. NOL = rem. fig. C:
and ∵ AE = EB,
∴ □ AN = □ NB, i.e. BM ;36 and 43.1.
add to ea. □ NO,
∴ whl. □ AX = gno. NOL;
but NOL = fig. C,
∴ □ AX = fig. C.

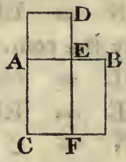
Therefore, to the rt. line AB is applied a □ AX = rectilin. fig. C, and exceeding by □ PO simil. □ D, for PO simil. EL.*

Q. E. D. *24.6.

PROP. XXX.—PROBLEM.

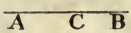
To cut a given right line in extreme and mean ratio.

Let AB be the given rt. line; it is required to cut it in extreme and mean ratio.



On AB descr. sq. BC; 46. 1.
to AC apply a □ CD = sq. BC, 29. 6.
and exceeding by a fig. AD simil. fig. BC.
But BC is a sq.
∴ AD is a sq.
and ∵ sq. BC = □ CD, constr.
take from ea. the com. □ CE,
∴ rem. □ BF = rem. □ AD;
and □s BF, AD are equiang.
∴ their sides about = ∠s are recip. propor.
i.e. FE : ED : : AE : EB.
Now FE = AC, i.e. AB, 34. 1.
and ED = AE,
∴ BA : AE : : AE : EB;
but AB > AE,
∴ AE > EB. 14. 5.
∴ AB is cut in extreme and mean ratio in E. 3 def. 6.

Q. E. F.



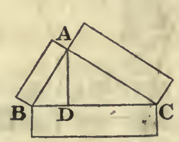
Otherwise
divide AB in C,
so that AB × BC = AC^2. 11. 2.
Then ∵ AB × BC = AC^2,
∴ BA : AC : : AC : CB. 17. 6.
∴ AB is cut in extreme and mean ratio in C. 3 def. 6.

Q. E. F.

PROP. XXXI.—Theorem.

In right angled triangles, the rectilineal figure described upon the side opposite to the right angle, is equal to the similar and similarly described figures upon the sides containing the right angle.

Let ABC be a rt. ∠d △, having rt. ∠ BAC; the rectilin. fig. described upon BC = the simil. and similarly described figs. upon BA, AC.



Draw AD ⊥ BC.

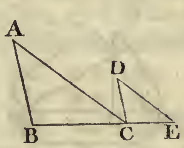
Then, ∵ in △ ABC, AD is drawn from rt. ∠ A ⊥ base BC,
∴ △s ABD, ADC simil. △ ABC and each other: 8. 6.
and ∵ △ ABC simil. △ ADB,
∴ CB : BA : : BA : BD; 4. 6.
and ∴ CB : BD : : fig. descr. on CB : simil. and similarly descr. fig. on BA; 2. cor. 20. 6.
and ∴ invert. DB : BC : : fig. on BA : fig. on BC: B. 5.
similarly DC : CB : : fig. on CA : fig. on CB:
∴ BD + DC : BC : : figs. on BA & AC : fig. on BC; 24. 5.
but BD + DC = BC,
∴ fig. descr. on BC is = to the simil. and similarly descr. figs. on BA, AC.

Wherefore, in right angled triangles, &c. &c. Q. E. D.

PROP. XXXII.—Theorem.

If two triangles which have two sides of the one proportional to two sides of the other, be joined at one angle so as to have their homologous sides parallel to one another; the remaining sides shall be in a right line.

Let ABC, DCE be two △s which have the two sides BA, AC propor. to the two CD, DE, i.e. BA : AC :: CD : DE; and let AB ∥ CD, and AC ∥ DE. Then BC, CE are in a rt. line.



∵ AC falls on ∥s AB, DC,
∴ ∠BAC = ∠ACD : 29. 1.
similarly ∠CDE = ∠ACD;
and ∴ ∠BAC = ∠CDE :
and ∵ in △ ABC, ∠ at A = ∠D in △ DCE,
and that the sides about these = ∠s are propors.
i.e. BA : AC :: CD : DE,
∴ △ABC is equiang. to △DCE; 6. 6.
and ∴ ∠ABC = ∠DCE :
now ∠BAC = ∠ACD, demon.
∴ whl. ∠ACE = ∠ABC + BAC;
add com. ∠ACB,
∴ ∠s ACE + ACB = ∠s ABC + BAC + ACB;
but ∠s ABC + BAC + ACB = 2 rt. ∠s, 32. 1.
∴ ∠ACE + ∠ACB = 2 rt. ∠s :
now, ∵ at C, in AC, on opp. sides of AC, BC, CE, make adjac. ∠s = 2 rt. ∠s,
∴ BC and CE are in one rt. line. 14. 1.

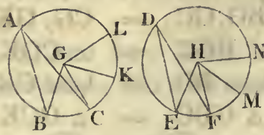
Therefore, if two triangles, &c. &c. Q. E. D.

PROP. XXXIII.—Theorem.

In equal circles, angles, whether at the centres or circumferences, have the same ratios which the arcs on which they stand have to each other; So also have the sectors.

Let ABC, DEF be equal ⊙s; and at their cents. the ∠s BGC, EHF, and the ∠s BAC, EDF at their ○s; then

First—$\overset{\frown}{BC} : \overset{\frown}{EF} :: \angle BGC : \angle EHF :: \angle BAC : EDF$.



Take any number of arcs,

viz. $\begin{cases} \overset{\frown}{CK}, \overset{\frown}{KL} \text{ ea.} = \overset{\frown}{BC}, \\ \text{and } \overset{\frown}{FM}, \overset{\frown}{MN} \text{ ea.} = \overset{\frown}{EF}; \end{cases}$

join GK, GL; HM, HN.

And ∵ $\overset{\frown}{BC}, \overset{\frown}{CK}, \overset{\frown}{KL}$ = ea. other,

∴ ∠s BGC, CGK, KGL = ea. other; 27. 1.

and ∴ ∠ BGL is same mult. of ∠ BGC that $\overset{\frown}{BL}$ is of $\overset{\frown}{BC}$:

similarly ∠ EHN is same mult. of ∠ EHF that $\overset{\frown}{EN}$ is of $\overset{\frown}{EF}$:

and if $\overset{\frown}{BL} = \overset{\frown}{EN}$,

then ∠ BGL = ∠ EHN;

and if greater, greater; if less, less.

Now, ∵ there are four mags. $\overset{\frown}{BC}$, $\overset{\frown}{EF}$, and ∠ BGC and ∠ EHF, and that of $\overset{\frown}{BC}$ and ∠ BGC are taken any equimults. $\overset{\frown}{BL}$ and ∠ BGL,

and also of $\overset{\frown}{EF}$ and ∠ EHF are taken any equimults. $\overset{\frown}{EN}$ and ∠ EHN,

and that if $\overset{\frown}{BL} > \overset{\frown}{EN}$,

then ∠ BGL > ∠ EHN,

and if equal, equal; if less, less.

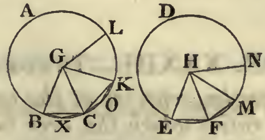
∴ $\overset{\frown}{BC} : \overset{\frown}{EF} :: \angle BGC : \angle EHF$; 5 def. 5.

but ∠ BGC : ∠ EHF :: ∠ BAC : ∠ EDF, 15. 6.

(for each is double of each,) 20. 3.

∴ $\overset{\frown}{BC} : \overset{\frown}{EF} :: \angle BGC : \angle EHF :: \angle BAC : \angle EDF$.

PROP. XXXIII. CONTINUED.



SECONDLY—Also $\overset{\frown}{BC} : \overset{\frown}{EF} :: \text{sect. } BGC : \text{sect. } EHF$.

Join BC, CK;

in $\overset{\frown}{BC}$, $\overset{\frown}{CK}$ take any pts. X and O;

join BX, XC, CO, OK.

Then, ∵ in △ GBC; BG, GC = CG, GK; in △ GCK,

and that ∠ BGC = ∠ CGK,

∴ base BC = base CK, } 4. 1.

and △ GBC = △ GCK: }

and ∵ $\overset{\frown}{BC} = \overset{\frown}{CK}$,

∴ rem. of whl. ○ of ⊙ ABC = rem. of whl. ○ of same ⊙;

∴ ∠ BXC = ∠ COK; 27. 3.

and ∴ seg. BXC simil. seg. COK: 11 def. 3.

and ∵ they are on equal rt. lines,

∴ seg. BXC = seg. COK; 24. 3.

and △ BGC = △ CGK,

∴ whl. sect. BGC = whl. sect. CGK;

and similarly sect. KGL = ea. of the sects. BGC, CGK:

and similarly it may be proved, that sects. EHF, FHM, MHN = ea. other.

∴ sect. BGL is same mult. of sect. BGC that $\overset{\frown}{BL}$ is of $\overset{\frown}{BC}$;

also sect. EHN is same mult. of sect. EHF that $\overset{\frown}{EN}$ is of $\overset{\frown}{EF}$,

and if $\overset{\frown}{BL} = \overset{\frown}{EN}$,

then sect. BGL = sect. EHN;

if greater, greater; if less, less.

Now, ∵ there are four mags. $\overset{\frown}{BC}$, $\overset{\frown}{EF}$, and sects. BGC and EHF,

and that of $\overset{\frown}{BC}$ and BGC are taken any equimults. $\overset{\frown}{BL}$, BGL,

also of $\overset{\frown}{EF}$ and EHF are taken any equimults. $\overset{\frown}{EN}$, EHN,

and that if $\overset{\frown}{BL} > \overset{\frown}{EN}$,

then sect. BGL > sect. EHN,

if equal, equal; and if less, less,

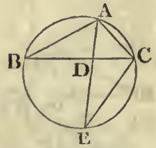
∴ $\overset{\frown}{BC} : \overset{\frown}{EF} :: \text{sect. } BGC : \text{sect. } EHF$.

Wherefore in equal circles, &c. &c. Q. E. D.

PROP. B.—Theorem.

If an angle of a triangle be bisected by a right line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the right line bisecting the angle.

Let ABC be a △, and let ∠ BAC be bisected by the rt. line AD; then $BA \times AC = BD \times DC + AD^2$.



About △ ABC descr. ⊙ ACB; 5.4.

prod. AD to E in ○;

join EC.

Then, ∵ ∠ BAD = ∠ CAE,

and that ∠ ABD = ∠ AEC, 21.3.

(for they are in same seg.;)

∴ △s ABD, AEC are equiang. to ea. other;

∴ BA : AD : : EA : AC; 4.6.

and consequently $BA \times AC = EA \times AD$; 16.6.

i. e. $BA \times AC = ED \times DA + AD^2$; 3.2.

but $ED \times DA = BD \times DC$, 35.3.

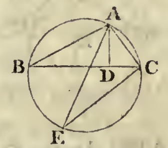
∴ $BA \times AC = BD \times DC + AD^2$.

Wherefore, if an angle, &c. &c. Q. E. D.

PROP. C.—Theorem.

If from any angle of a triangle a right line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a △, and AD the ⊥ from ∠ at A, to the base BC; then BA × AC = AD × diam. of the ⊙ descr. about the △.



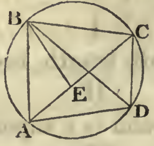
About △ ABC descr. ⊙ ACB; 5. 4.
draw its dia. AE;
join EC.
Then ∵ rt. ∠ BDA = ECA in a ½ ⊙, 31. 3.
and ∠ ABD = ∠ AEC in same seg. 21. 3.
∴ △s ABD, AEC are equiang.
∴ BA : AD : : EA : AC; 4. 6.
and ∴ BA × AC = EA × AD.

Therefore, if from any angle, &c. &c. Q. E. D.

PROP. D.—Theorem.

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles together, contained by its opposite sides.

Let ABCD be any quadrilat. inscribed in a ⊙, and join AC, BD its diags.; then $AC \times BD = AB \times CD + AD \times BC$.



Make $\angle ABE = \angle DBC$;
add to ea. the com. $\angle EBD$,
$\therefore \angle ABD = \angle EBC$:
and $\angle BDA = \angle BCE$ in same seg. 21.3.
$\therefore \triangle$s ABD, BCE are equiang.
$\therefore BC : CE :: BD : DA$; 4.6.
and $\therefore BC \times AD = BD \times CE$. 16.6.
Again, $\because \angle ABE = \angle DBC$,
and $\angle BAC = \angle BDC$, 21.3.
$\therefore \triangle$s ABE, BCD are equiang.
$\therefore BA : AE :: BD : DC$;
and $\therefore BA \times DC = BD \times AE$;
but $BC \times AD = BD \times CE$,
$\therefore$ whl. $AC \times BD = AB \times CD + AD \times BC$.

Wherefore the rectangle, &c. &c. Q. E. D.

BOOK XI.

DEFINITIONS.

I.

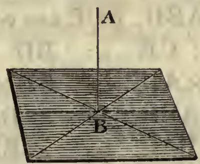
A SOLID is that which hath length, breadth, and thickness.

II.

That which bounds a solid is a superficies.

III.

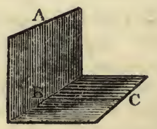
A right line is perpendicular, or at right angles, to a plane, when it makes right angles with every right line in that plane which meets it.



IV.

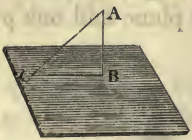
A plane is perpendicular to a plane, when the right lines drawn in one of the planes perpendicular to the common section of the two planes, are perpendicular to the other plane.

Thus the plane in which the right line AB is drawn is perpendicular to the plane in which right line BC is drawn, for AB is at right angles to BC.



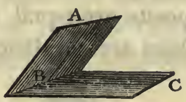
V.

The inclination of a right line to a plane, is the acute angle contained by that right line, and another drawn from the point in which the first line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the first line above the plane, meets the same plane.



VI.

The inclination of a plane to a plane is the acute angle contained by two right lines drawn from any the same point of their common section at right angles to it, one upon one plane, and the other upon the other plane.



VII.

Two planes are said to have the same or a like inclination to each other which two other planes have, when the said angles of inclination are equal to each other.

VIII.

Parallel planes are such as do not meet each other though produced.

IX.

A solid angle is that which is made by the meeting of more than two plane angles, which are not in the same plane, in one point.

X.

Equal and similar solid figures are such as are contained under an equal number of equal and similar planes.*

* Dr. Simson has omitted this definition altogether. He says, that it is properly a theorem, and requires demonstration. And therefore accuses Theon of the interpolation.

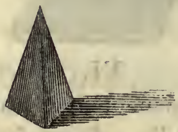
That figures are similar, he observes, ought to be proved from the definitions of similar figures; and that they are equal ought to be demonstrated from the axiom, "Magnitudes that wholly coincide, are equal;" or

XI.

Similar solid figures are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.

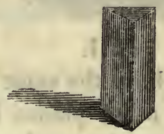
XII.

A pyramid is a solid figure contained by planes that are constituted betwixt one plane and one point above it in which they meet.



XIII.

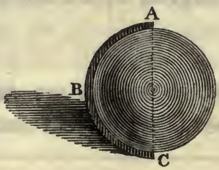
A prism is a solid figure contained by plane figures, of which, two that are opposite are equal, similar, and parallel to each other; and the others are parallelograms.



XIV.

A sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved.

Thus the inner side of the semicircle ABC revolving round the diameter AC, which remains fixed, generates a sphere.



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or from props. A or 9th or 14th of 5th Book, from one of which the equality of all kinds of figures must be ultimately deduced.

The propositions A, B, C, are added to supply this and other defects.

XV.

The axis of a sphere is the fixed right line about which the semicircle revolves.

Thus AC, in the figure above, is the axis of the sphere.

XVI.

The centre of a sphere is the same with that of the semicircle.

XVII.

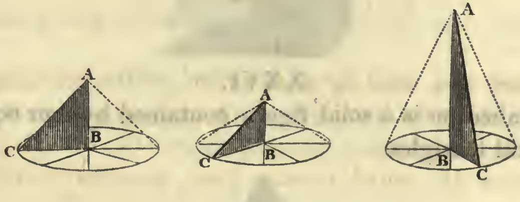
The diameter of a sphere is any right line which passes through the centre, and is terminated both ways by the superficies of the sphere.

XVIII.

A cone is a solid figure described by the revolution of a right angled triangle about one of the sides containing the right angle, which side remains fixed.

If the fixed side be equal to the other side containing the right angle, the cone is called a right angled cone; if it be less than the other side, an obtuse angled; and if greater, an acute angled cone.

Thus the side AC, revolving round AB, one of the sides which contains the right angle and remains fixed, generates a cone.



XIX.

The axis of a cone is the fixed right line about which the triangle revolves.

In fig. above, AB is the axis.

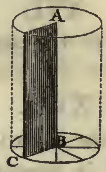
XX.

The base of a cone is the circle described by that side containing the right angle which revolves.

XXI.

A cylinder is a solid figure described by the revolution of a right angled parallelogram about one of its sides which remains fixed.

Thus the revolution of the parallelogram AC about its side AB, which remains fixed, generates a cylinder.



XXII.

The axis of a cylinder is the fixed right line about which the parallelogram revolves.

XXIII.

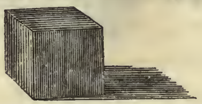
The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.

XXIV.

Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

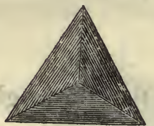
XXV.

A cube is a solid figure contained by six equal squares.



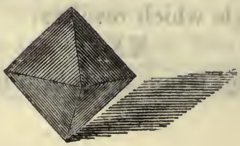
XXVI.

A tetrahedron is a solid figure contained by four equal and equilateral triangles.



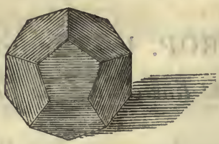
XXVII.

An octahedron is a solid figure contained by eight equal and equilateral triangles.



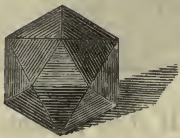
XXVIII.

A dodecahedron is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.



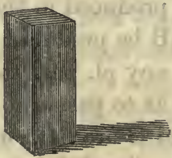
XXIX.

An icosahedron is a solid figure contained by twenty equal and equilateral triangles.



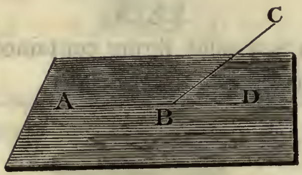
Def. A.

A parallelopiped is a solid figure contained by six quadrilateral figures, whereof every opposite two are parallel.



PROP. I.—Theorem.

One part of a right line cannot be in a plane and another part above it.



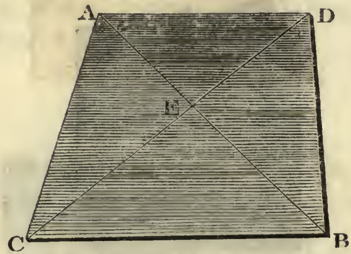
If it be possible,
let AB, part of rt. line ABC, be in the plane,
and the part BC elevated above the plane.
And ∵ AB is in a plane,
it can be produced in that plane.
Let AB be produced to D.
And let any pl. pass thro. AD,
and so as to pass thro. pt. C.
Then ∵ pts. B and C are both in the same plane,
∴ rt. line BC is in it. 7 def. 1.
∴ There are two rt. lines ABC, ABD, in same pl.
which have a com. seg. AB;
which is impossible. cor. 11.1.

Therefore one part, &c. &c. Q. E. D.

PROP. II.—Theorem.

Two right lines which cut each other are in one plane, and three right lines which meet each other are in one plane.

Let two rt. lines AB, CD cut each other in E; AB, CD are in one plane. And the three rt. lines EC, CB, BE which meet ea. other, are in one plane.



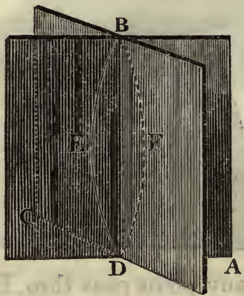
Let any plane pass thro. EB;
and let it be turned about EB,
and produced, if necessary, until it pass thro. C.
Then ∵ E and C are in same plane,
∴ rt. line EC is in the plane. 7 def. 11.
Similarly BC is in the same plane;
but by hyp. EB is in the plane,
∴ EC, CB, BE are in one plane.
Now CD, AB, are in same plane with EC, EB. 1. 11.
∴ AB, CD are in one plane.

Wherefore, right lines, &c. &c. Q. E. D.

PROP. III.—Theorem.

If two planes cut each other, their common section is a right line.

Let plane AB cut the plane BC; and let DB be their common section, then DB is a rt. line.



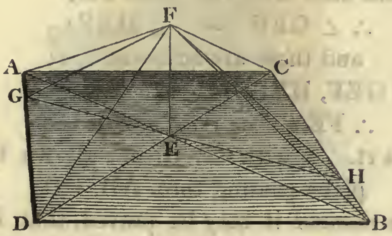
If not,
from D to B, draw rt. line DEB in the pl. AB;
and from D to B, draw rt. line DFB in the pl. BC:
consequently DEB, DFB have the same extrems.;
and ∴ the rt. lines DEB, DFB inclose a space;
which is impossible. 10 ax. 1.
∴ BD the com. sect. of planes AB, BC is a rt. line.

Wherefore if two planes, &c. &c. Q. E. D.

PROP. IV.—Theorem.

If a right line stand at right angles to each of two right lines in the point of their intersection, it shall also be at right angles to the plane which passes through them, that is, to the plane in which they are.

Let the rt. line EF stand at rt. ∠s to ea. of rt. lines AB, CD in E, the pt. of their intersec.: EF is also at rt. ∠s to the plane passing thro. AB, CD.



Take rt. lines AE, EB, CE, ED = ea. other;
thro. E, draw GEH in the pl. in which are AB, DC;
join AD, CB;
from any pt. F in EF, draw FA, FG, FD, FB, FH, FC.

And ∵ AE, ED	=	BE, EC ea. to ea.,	
and that ∠ AED	=	∠ BEC,	15. 1.
∴ base AD	=	base BC, }	4. 1.
and ∠ DAE	=	∠ EBC: }	
and ∠ AEG	=	∠ BEH,	15. 1.
∴ in △ AEG; ∠sGAE,AEG	=	∠s EBH, HEB in △ BEH;	

also *sides* adjac. to equal ∠s are = *ea. other*,

i. e. AE	=	EB;	
and ∴ also GE	=	EH, }	26. 1.
and AG	=	BH: }	
and ∵ AE	=	EB,	

and that EF is com. and at rt. ∠s to them,

∴ base AF	=	base FB:	4. 1.

 similarly

PROP. IV. CONTINUED.

similarly CF = FD;
and ∵ AD = BC,
and AF = FB,
and that base DF = base FC,
∴ ∠ FAD = ∠ FBC. 8. 1.
Again, ∵ GA = BH, demon.
and AF = FB,
and that ∠ FAG = ∠ FBH,
∴ base FG = base FH. 4. 1.
Again, ∵ GE = EH, demon.
and EF is com.
and that base GF = base FH,
∴ ∠ GEF = ∠ HEF:
and these are adjacent ∠ s;
∴ ea. of ∠ s GEF, HEF is a rt. ∠: 10 def. 1.
∴ FE makes rt. ∠ s with GH;
i. e. FE makes rt. ∠ s with any rt. line drawn thro. E in the plane passing thro. AB, CD.

In the same manner it may be proved, that FE makes rt. ∠ s with every rt. line which meets it in that plane. Now a rt. line is at rt. ∠ s to a plane, when it makes rt. ∠ s with every rt. line which meets it in that plane.* * 3 def. 11.

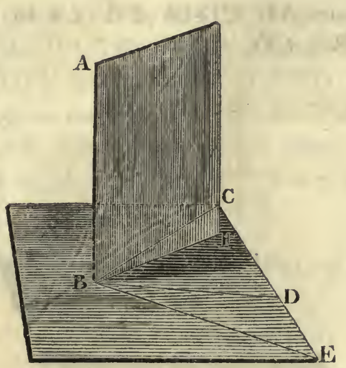
∴ EF is at rt. ∠ s to plane passing thro. AB, CD.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. V.—Theorem.

If three right lines meet all in one point, and a right line stands at right angles to each of them in that point; these three right lines are in one and the same plane.

Let the rt. line AB stand at rt. ∠s to ea. of the rt. lines BC, BD, BE, in B the pt. where they meet. BC, BD, BE are in one and the same plane.



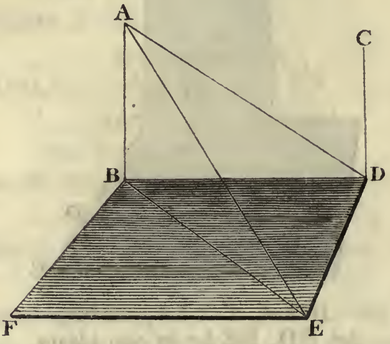
If not, if it be possible,
let BD, BE be in one plane,
and BC be elevated above it;
and let a plane pass thro. AB, BC;
then the sec. of this pl. with the pl. thro. BD, BC, is a rt. line: 3. 11.
let this rt. line be BF;
∴ AB, BC, BF are in one plane;
viz. in that which passes thro. AB, BC.
Now ∵ AB is rt. ∠s to BD and BE,
∴ AB is rt. ∠s to plane thro. BD, BE; 4. 11.
and ∴ AB is rt. ∠s to every rt. line meeting it in that plane; 3 def. 11.
Now BF, which is in that plane, meets AB,
∴ ∠ ABF is a rt. ∠;
but ∠ ABC is a rt. ∠, hyp.
∴ ∠ ABF = ∠ ABC;
and they are both in same plane;
which is impossible;
∴ BC is not above the plane in which are BD, BE;
i. e. BC, BD, BE are in one and the same plane.

Wherefore if three right lines, &c. &c. Q. E. D.

PROP. VI.—THEOREM.

If two right lines be at right angles to the same plane, they shall be parallel to each other.

Let the rt. lines AB, CD be at rt. ∠s to the same plane EF; then is AB ∥ CD.



Let AB, CD meet the plane in B, D;
draw rt. line BD;
draw DE rt. ∠s to BD in same plane FD;
make DE = AB;
join BE, AE, AD.
Then, ∵ AB ⊥ plane FD,
∴ AB is rt. ∠s to every rt. line which meets it in FD; 3 def.11.
now BD, BE, which are in FD, meet AB,
∴ ea. of the ∠s ABD, ABE is a rt. ∠:
and similarly ea. of the ∠s CDB, CDE is a rt. ∠.
And ∵ AB = DE,
and BD is com.
and that rt. ∠ ABD = rt. ∠ BDE,
∴ base AD = base BE. 4. 1.
Again, ∵ AB = DE,
and that BE = AD,

and

PROP. VI. CONTINUED.

and base AE is com. to △s ABE, EDA,

∴ ∠ ABE = ∠ EDA: 8. 1.

but ∠ ABE is a rt. ∠,

∴ ∠ EDA is a rt. ∠;

and conseq. ED ⊥ DA;

but also ED ⊥ BD and DC,

∴ ED is rt. ∠s to ea. of BD, DA, DC in pt. where they meet,

∴ BD, DA, DC are in one plane BC: 5. 11.

now AB is in same plane with BD, DA,

(for any three rt. lines meeting ea. other are in one plane,) 2. 11.

∴ AB, BD, DC are in one plane;

and ea. of ∠s ABD, BDC is a rt. ∠,

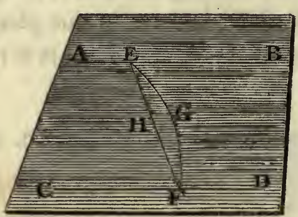
∴ AB ∥ CD. 28. 1.

Wherefore, if two rt. lines, &c. &c. Q. E. D.

PROP. VII.—THEOREM.

If two right lines be parallel, the right line drawn from any point in the one to any point in the other, is in the same plane with the parallels.

Let AB, CD be ∥ rt. lines, and take any pts. E in AB and F in CD. The rt. line which joins E and F are in the same plane AD with the ∥ s.



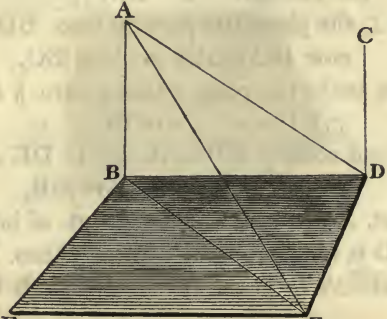
If not, if it be possible,
let it be above the plane AD, as EGF:
and in plane AD draw EHF from E to F:
and ∵ EGF is also a rt. line,
∴ EGF, EHF include a space;
which is impossible: 10 ax. I.
∴ the rt. line joining pts. E, F is not above the plane AD,
i. e. it is in the same plane with AB, CD.

Wherefore, if two rt. lines, &c. &c. Q. E. D.

PROP. VIII.—Theorem.

If two right lines be parallel, and one of them is at right angles to a plane; the other also shall be at right angles to the same plane.

Let AB, CD be ∥ rt. lines, and let one, AB, be at rt. ∠s to plane FD; then CD is at rt. ∠s to the same plane.



Let AB, CD meet the plane FD in B, D;
join BD;
∴ AB, CD, BD are in one plane BC: 7. 11.
in plane FD, draw DE rt ∠s. to BD;
and make DE = AB;
join BE, AE, AD:
then, ∵ AB ⊥ plane FD,
∴ AB ⊥ BD, BE; 3 def. 11.
∴ ∠ABD or ∠ABE is a rt. ∠:
and ∵ BD meets ∥ s AB, CD,
∴ ∠ABD + ∠CDB = 2 rt. ∠s; 29. 1.
but ∠ABD is a rt. ∠,
∴ ∠CDB is a rt. ∠;
and ∴ CD ⊥ BD:
and ∵ AB = DE,
and BD is com.

and

PROP. VIII. CONTINUED.

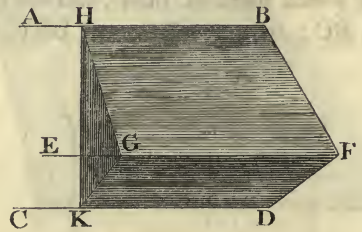
and that rt. ∠ ABD = rt. ∠ EDB,
∴ base AD = base BE. 4. 1.
Again, ∵ AB = DE,
and BE = AD,
and that base AE is com. to △ s ABE, EDA,
∴ ∠ ABE = ∠ EDA ; 8. 1.
but ∠ ABE is a rt. ∠,
∴ ∠ EDA is a rt. ∠ ;
and ∴ ED ⊥ DA ;
but also ED ⊥ BD,
∴ ED ⊥ the plane BC passing thro. BD, DA : 4. 11.
now DC is also in plane BC,
(for all these are in the plane passing thro. ∥ s AB, CD,)
∴ ED is rt. ∠ s to DC 3 def. 11.
and conseq. CD is rt. ∠ s to DE ;
but also CD is rt. ∠ s to DB,
∴ CD is rt. ∠ s to DE, and DB in pt. of intersec. D ;
and ∴ CD is rt. ∠ s to plane passing thro. DE, DB ;
i.e. CD is rt. ∠ s to plane FD to which AB is at rt. ∠ s.

Wherefore, if two right lines, &c. &c. Q. E. D.

PROP. IX.—Theorem.

Two right lines which are each of them parallel to the same right line, and not in the same plane with it, are parallel to each other.

Let AB, CD be ea. ‖ EF, and not in same plane with it; AB shall be ‖ CD.



In EF take any pt. G;
in plane EB, passing thro. AB, EF,
draw from G, GH at rt. ∠ s to EF;
and in plane ED passing thro. EF, CD,
draw from G, GK at rt. ∠ s to EF:

and ∵ EF	⊥	GH, and GK,	
∴ EF	⊥	pl. HGK thro. GH, GK:	4. 11.
Now EF	‖	AB,	
∴ AB	⊥	pl. HGK:	8. 11.
similarly CD	⊥	pl. HGK,	

∴ AB and CD are ea. rt. ∠ s to pl. HGK,

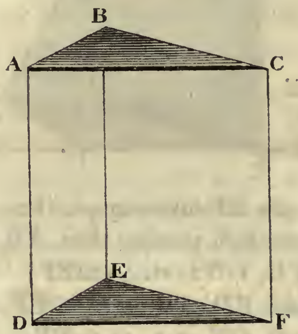
∴ AB	‖	CD.	6. 11.

Wherefore, two right lines, &c. &c. Q. E. D.

PROP. X.—Theorem.

If two right lines meeting each other be parallel to two others which meet each other, and are not in the same plane with the first two; the first two and the other two shall contain equal angles.

Let the two rt. lines AB, BC, which meet ea. other, be ∥ to the two DE, EF which meet ea. other, and are not in the same plane with AB, BC; then ∠ ABC = ∠ DEF.



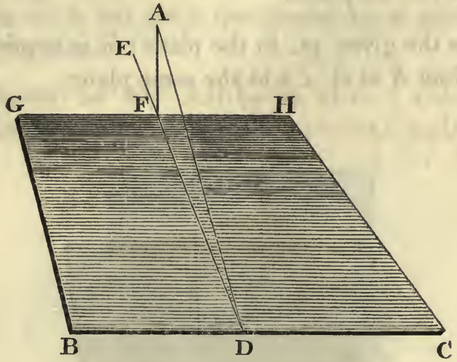
Take AB, BC, DE, EF = ea. other;
join AD, BE, CF, AC, DF.
Then, ∵ AB = and ∥ DE,
∴ AD = and ∥ BE: 33. 1.
Similarly CF = and ∥ BE,
and ∴ AD = and ∥ CF: 9.11. and 1. ax. 1.
now AC, DF join AD, CF toward same parts,
∴ AC = and ∥ DF: 33. 1.
and ∵ AB, BC = DE, EF ea. to ea.
and base AC = base DF,
∴ ∠ ABC = ∠ DEF. 8. 1.

Therefore, if two right lines, &c. &c. Q. E. D.

PROP. XI.—Problem.

To draw a right line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH: it is required to draw from A a rt. line ⊥ plane BH.



In plane BH draw any rt. line BC;
from A draw AD ⊥ BC;
then, if AD ⊥ plane GH,
the thing required is done.
But, if not;
in plane BH, draw from D, DE rt. ∠s to BC;
and from A draw AF ⊥ DE;
and thro. F draw GH ∥ BC;
and ∵ BC is rt. ∠s to ED, and DA,
∴ BC is rt. ∠s to plane passing thro. ED, DA: 4. 11.
and ∵ GH ∥ BC,
∴ GH is rt. ∠s to plane passing thro. ED, DA: 8. 11.
and ∵ AF, in same pl. with ED, DA, meets GH,
∴ GH ⊥ AF; 3 def. 11.
and conseq. AF ⊥ GH;
but AF ⊥ DE,
∴ AF ⊥ GH, & DE, in pt. of inters. F;
∴ AF is rt. ∠s to plane passing thro. GH, DE: 4. 11.
now BH is that plane.
∴ AF ⊥ plane BH.

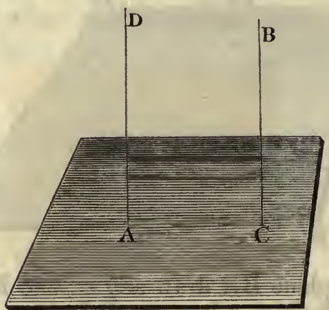
Therefore, from pt. A, a rt. line AF is drawn ⊥ plane BH.

Q. E. F.

PROP. XII.—Problem.

To erect a right line at right angles to a given plane, from a point given in the plane.

Let A be the given pt. in the plane; it is required to erect a rt. line from A at rt. $\angle$s to the same plane.



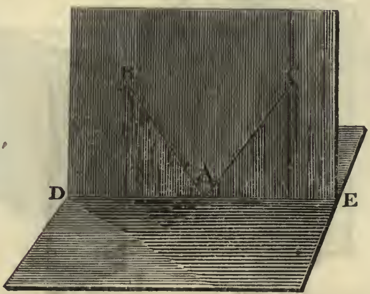
From any pt. B above the plane, } 11.11.
draw BC $\perp$ to the plane; }
from A, draw AD $\parallel$ BC.
Then, $\because$ AD, CB are two $\parallel$ rt. lines,
and that one BC is rt. $\angle$s to given plane,
$\therefore$ AD is rt. $\angle$s to same plane. 8. 11.

Therefore, rt. line AD has been erected from pt. A, in the given plane, $\perp$ to that plane. Q. E. F.

PROP. XIII.—Theorem.

From the same point in a given plane, there cannot be two right lines at right angles to the plane, upon the same side of it: and there can be but one perpendicular to a plane from a point above the plane.

For if possible, let AC, AB be ea. at rt. ∠s to the given plane, from one pt. A in same plane and on the same side of it.



Let a pl. pass thro. BA, AC;
then the com. sec. of the two planes is a rt. line. 3. 11.
Let DAE be their common sec.;
∴ AB, AC, DAE are in one plane:
and ∵ AC is rt. ∠s to given plane,
and that rt. line DAE meets AC in that plane,
∴ ∠ CAE is a rt. ∠: 3 def. 11.
similarly ∠ BAE is a rt. ∠,
∴ ∠ CAE = ∠ BAE;
and they are in one plane,
which is impossible.

Also from a pt. above a plane, there can be but one perpendicular to that plane; for, if there could be two, they would be ∥ ea. other,* *6. 11.

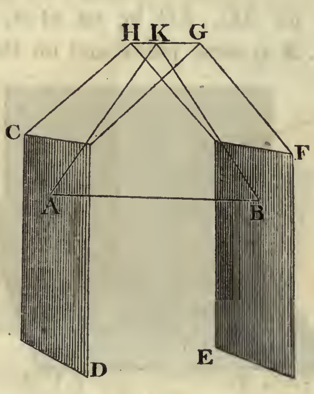
which is absurd.

Therefore, from the same point, &c. &c. Q. E. D.

PROP. XIV.—Theorem.

Planes to which the same right line is perpendicular, are parallel to each other.

Let rt. line AB be ⊥ to ea. of the planes CD, EF; then the planes are ‖ to ea. other.



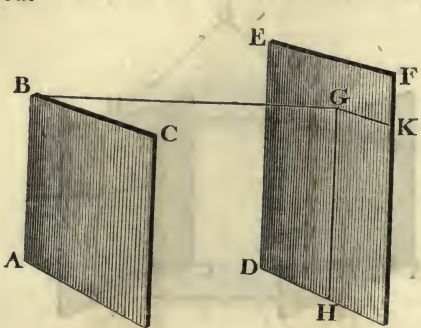
If not,
they shall meet when produced,
and their sec. shall be a rt. line GH;
in GH take any pt. K;
join AK, BK.
Then, ∵ AB ⊥ plane EF,
∴ AB ⊥ rt.line BK in that pl.; 3 def. 11.
and ∴ ∠ABK is a rt. ∠:
similarly ∠BAK is a rt. ∠,
∴ two ∠s ABK, BAK of one △ ABK = 2 rt. ∠s,
which is impossible. 17. 1.
∴ The planes CD, EF being prod. do not meet;
i. e. pls. CD, EF ‖ ea. other.

Wherefore planes, &c. &c. Q. E. D.

PROP. XV.—Theorem.

If two right lines meeting each other, be parallel to two other lines which meet, but are not in the same plane with the first two; the plane which passes through these is parallel to the plane passing through the others.

Let AB, BC, two rt. lines meeting each other, be $\parallel$ to DE, EF which meet, but are not in same plane with AB, BC. Then the planes thro. AB, BC, and DE, EF shall not meet, tho. produced.



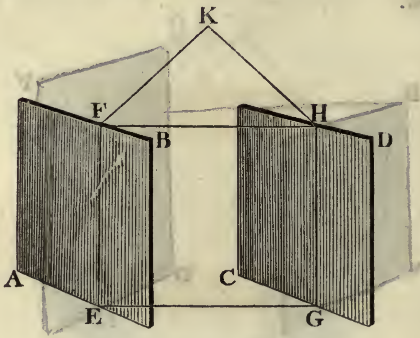
From B, draw BG $\perp$ pl. DF thro. DE, EF;
and let BG meet DF in G;
thro. G, draw $\begin{cases} \text{GH} \parallel \text{ED;} \\ \text{and GK} \parallel \text{EF:} \end{cases}$
and $\because$ BG $\perp$ plane DF,
and that GH, GK meet BG in that plane,
$\therefore$ BG is rt. $\angle$s to GH and GK; 3 def. 11.
$\therefore$ $\angle$ BGH or $\angle$ BGK is a rt. $\angle$.
And $\because$ BA $\parallel$ GH, 9. 11.
(for ea. of them is $\parallel$ DE and not in same plane with it),
$\therefore$ $\angle$ GBA + $\angle$ BGH = 2 rt. $\angle$s: 29. 1.
now $\angle$ BGH is a rt. $\angle$,
$\therefore$ $\angle$ GBA is a rt. $\angle$;
and $\therefore$ GB $\perp$ BA:
similarly GB $\perp$ BC:
and $\because$ GB is rt. $\angle$s to rt. lines BA, BC in pt. of intersec. B,
$\therefore$ GB $\perp$ plane AC; 4. 11.
but also GB $\perp$ plane EF,
$\therefore$ pl. thro. AB, BC $\parallel$ pl. thro. DE, EF. 14. 11.

Wherefore if two right lines, &c. &c. Q. E. D.

PROP. XVI.—Theorem.

If two parallel planes be cut by another plane, their common sections with it are parallels.

Let the two parallel planes AB, CD be cut by the plane EH; and let their secs. with it be EF, GH: then EF ∥ GH.



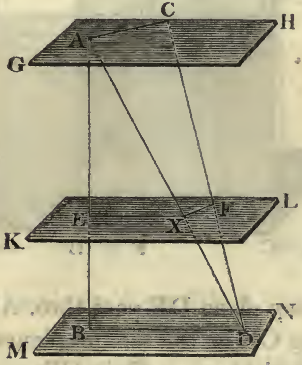
For if EF be not ∥ GH,
then EF, GH will meet, if prod. either on the side of FH or EG.
First—Let EF, GH meet, on the side of FH, in K.
And ∵ rt. line EFK is in the plane AB,
∴ every pt. in EFK is in that plane;
but K is a pt. in EFK,
∴ K is in the plane AB;
similarly K is in the plane CD;
∴ AB, CD prod. will meet ea. other;
but AB ∥ CD, hyp.
∴ AB, CD do not meet ea. other;
∴ EF, GH do not meet if prod. on side of FH.
Secondly—In the same manner it may be demon.
that EF, GH do not meet if prod. on side of EG;
∴ EF ∥ GH. 35 def. 1.

Wherefore if two parallel planes, &c. &c. Q. E. D.

PROP. XVII.—Theorem.

If two right lines be cut by parallel planes, they shall be cut in the same ratio.

Let the rt. lines AB, CD be cut by the parallel planes GH, KL, MN, in the pts. A, E, B; C, F, D: then AE : EB :: CF : FD.



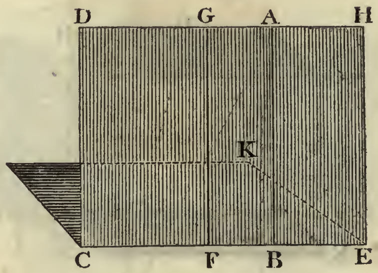
Join AC, BD, AD;
and let AD meet plane KL in X;
join EX, XF:
∵ paral. planes KL, MN are cut by plane BX,
∴ their com. secs. BD, EX are ∥ ea. other. 16.11.
Again, ∵ paral. planes KL, GH are cut by plane CX,
∴ their com. secs. AC, XF are ∥ ea. other.
Now, ∵ EX ∥ BD a side of △ ABD,
∴ AE : EB :: AX : XD. 2.6.
Again, ∵ XF ∥ AC a side of △ ADC,
∴ AX : XD :: CF : FD:
but AX : XD :: AE : EB, demon.
∴ AE : EB :: CF : FD. 11.5.

Wherefore if two right lines, &c. &c. Q. E. D.

PROP. XVIII.—Theorem.

If a right line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.

Let the right line AB be at rt. ∠s to a plane CK; then every plane which passes thro. AB shall be at rt. ∠s to plane CK.



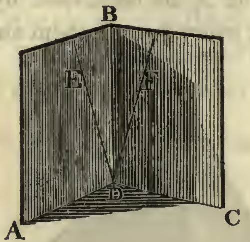
Let any plane DE pass thro. AB;
and let rt. line CE be the sec. of planes CK, DE;
take any pt. F, in CE;
from F draw $\overline{FG}$, in pl. DE, at rt. ∠s to CE.
And ∵ AB ⊥ plane CK,
∴ AB ⊥ CE; 3 def. 11.
and ∴ ∠ ABF is a rt. ∠;
but ∠ GFB is a rt. ∠,
∴ AB ∥ FG; 28. 1.
but AB is rt. ∠s to plane CK,
∴ FG is rt. ∠s to plane CK. 8. 11.
Now, ∵, in plane DE; $\overline{FG}$ ⊥ plane CK,
and that also it is rt. ∠s to CE the com. sec., constr.
∴ plane DE is rt. ∠s to plane CK. 4 def. 11.
similarly it may be demon. that all planes thro. AB are at rt. ∠s to plane CK.

Wherefore if a right line, &c. &c. Q. E. D.

PROP. XIX.—THEOREM.

If two planes which cut each other be each of them perpendicular to a third plane; their common section shall be perpendicular to the same plane.

Let the two planes AB, BC be ea. ⊥ to a third plane ADC, and let BD be the sec. of AB, BC. Then is BD ⊥ plane ADC.



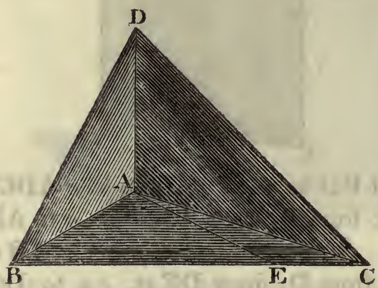
If BD be not ⊥ to plane ADC,
then in pl. AB, from D, draw DE rt. ∠s to AD sec. of pls. AB and ADC;
and in pl. BC, from D, draw DF rt. ∠s to DC sec. of pls. BC and ADC.
Now ∵ pl. AB ⊥ pl. ADC,
and that in AB. is drawn DE rt. ∠s to AD their com. sec.
∴ DE ⊥ pl. ADC: 4 def. 11.
similarlar DF ⊥ pl. ADC;
∴ from one pt. D, two rt. lines are rt. ∠s to a pl. ADC on one side of it,
which is impossible. 13. 11.
∴, from D, no rt. line can be drawn at rt. ∠s to plane ADC, except BD, the sec. of the two pls. AB, BC.
∴ BD ⊥ pl. ADC.

Wherefore, if two planes, &c. &c. Q. E. D.

PROP. XX.—Theorem.

If a solid angle be contained by three plane angles, any two of them are greater than the third.

Let the solid ∠ at A be contained by the three plane ∠s BAC, CAD, DAB, every two of them shall be > third.



If ∠s BAC, CAD, DAB = ea. other,
it is evident that any two together are > third:
but if they are ≠ ea. other;
let ∠ BAC be that which ≮ either of the others,
but > DAB.
Then in pl. passing thro. BA, AC, and at A, in AB,
make ∠ BAE = ∠ DAB; 23. 1.
and make AE = AD;
thro. E draw BEC cutting AB, AC in B and C;
join DB, DC.
Then, ∵ DA = AE,
and AB is com.
and that ∠ EAB = ∠ DAB,
∴ base DB = base BE: 4. 1.
and ∵ BD + DC > BC, 20. 1.

and

PROP. XX. CONTINUED.

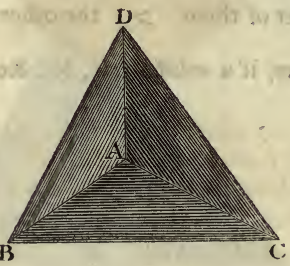
and that BD	$=$	BE part of BC,	
$\therefore$ DC	$>$	rem. part EC.	
Again, $\because$ DA	$=$	AE,	
and AC is com.			
and that base DC	$>$	base EC,	
$\therefore \angle$ DAC	$>$	$\angle$ EAC :	25. 1.
now $\angle$ DAB	$=$	$\angle$ BAE,	constr.
$\therefore \angle$ DAB $+ \angle$ DAC	$>$	$\angle$ BAE $+$ EAC ;	
i. e. $\angle$ DAB $+ \angle$ DAC	$>$	$\angle$ BAC ;	
but $\angle$ BAC	$\nless$	either of the $\angle$ s DAB, DAC,	
$\therefore \angle$ BAC $+$ either of them	$>$	the other.	

Wherefore, if a solid angle, &c. &c. Q. E. D.

PROP. XXI.—Theorem.

Every solid angle is contained by plane angles which together are less than four right angles.

First—Let the solid ∠ at A be contained by three plane ∠s BAC, CAD, DAB. Then these three together are < four rt. ∠s.



In AB, AC, AD take any pts. B, C, D;
join BC, CD, DB.

Then, ∵ sol. ∠ at B is cont. by three pl. ∠s CBA, ABD, DBC,

∴ any two of them > the third, 20.11.

∴ ∠CBA + ∠ABD > ∠DBC:

similarly $\left\{\begin{array}{l}\angle BCA + \angle ACD > \angle DCB; \\ \text{and } \angle CDA + \angle ADB > \angle BDC;\end{array}\right.$

∴ the 6 ∠s $\left\{\begin{array}{l}CBA,\ ABD,\ BCA \\ ACD,\ CDA,\ ADB\end{array}\right\} > 3\angle s \left\{\begin{array}{l}DBC,\ BCD, \\ CDB;\end{array}\right.$

but ∠s DBC + BCD + CDB = 2 rt. ∠s, 32. 1.

∴ the 6 ∠s $\left\{\begin{array}{l}CBA,\ ABD,\ BCA \\ ACD,\ CDA,\ ADB\end{array}\right\} > 2$ rt. ∠s:

now ∵ the 3 ∠s of ea. △ ABC, ACD, ADB = 2 rt. ∠s, 32. 1.

∴ whl. 9 ∠s $\left\{\begin{array}{l}CBA,\ BAC,\ ACB \\ ACD,\ CDA,\ DAC \\ ADB,\ DBA,\ BAD\end{array}\right\} = 6$ rt. ∠s;

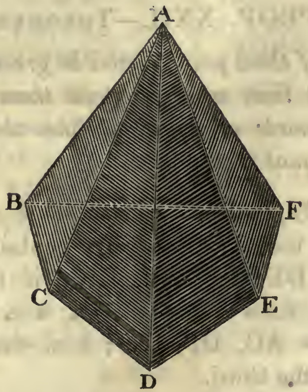
but 6 ∠s of these 9 are > 2 rt. ∠s; demon.

∴ rem. 3 ∠s BAC, DAC, BAD < 4 rt. ∠s.

Secondly,

PROP. XXI. CONTINUED.

SECONDLY—Let the solid ∠ at A be cont. by any number of plane ∠s BAC, CAD, DAE, EAF, FAB; these together shall be < 4 rt. ∠s.



Let the pls., in which the ∠s are, be cut by a pl. and let the secs. of it with these pls. be BC, CD, DE, EF, FB. Then ∵ sol. ∠ at B is cont. by 3 pl. ∠s CBA, ABF, FBC,

of which, any two are	>	third,
∴ ∠s ABC+ABF	>	∠ CBF:
similarly ∠s { ACD+ACB	>	∠ BCD,
ADE+ADC	>	∠ CDE,
AED+AEF	>	∠ DEF,
and AFE+AFB	>	∠ EFB:

but the ∠s { FBC, BCD, CDE, DEF and EFB } are the ∠s of fig. BCDEF,

∴ all the ∠s at bases of the △s	>	all the ∠s of the polyg.:
and ∵ all the ∠s of the △s together	=	2No. of rt. ∠s as there are △s, **32. 1.**
i. e.	=	2 No. of rt. ∠s as sides in fig.
and that all the ∠s of fig. + 4 rt. ∠s	=	2 No. of rt. ∠s as there are sides in fig. **1 cor. 32. 1.**
∴ all the ∠s of the △s together	=	all the ∠s of fig. + 4 rt. ∠s;
but all the ∠s at the bases of △s	>	all the ∠s of the fig. **demon.**

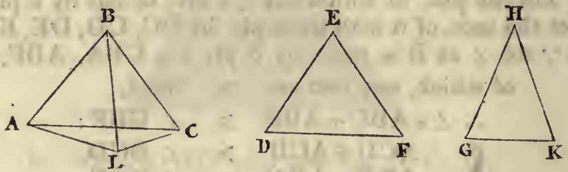
∴ rem. ∠s of the △s, which cont. sol. ∠ A < 4 rt. ∠s.

Therefore every solid angle, &c. &c. Q. E. D.

PROP. XXII.—Theorem.

If every two of three plane angles be greater than the third, and if the right lines which contain them be all equal; a triangle may be made of the right lines that join the extremities of those equal right lines.

Let ABC, DEF, GHK, be three plane $\angle$s, whereof every two are $>$ than the third, and are contained by the $=$ rt. lines AB, BC, DE, EF, GH, HK; if their extrems. be joined by the rt. lines AC, DF, GK, a $\triangle$ may be made of three rt. lines $=$ AC, DF, GK; i. e. every two of them shall be $>$ than the third.



If $\angle$s at B, E, H, $=$ each other,
then also AC, DF, GK $=$ each other; 4. 1.
and any two of them $>$ third:
but, if these $\angle$s $\neq$ each other;
let $\angle$ ABC $\not<$ $\angle$ E or $\angle$ H,
$\therefore$ AC $\not<$ DF, GK: 24. 1.
and $\therefore$ it is manifest,
that AC + either of them $>$ third.
Also DF + GK $>$ AC:
for, at B, in AB make $\angle$ ABL $=$ $\angle$ GHK; 23. 1.
and make BL $=$ either of AB, BC, DE, EF, GH, HK;
join AL, LC.
Then, $\because$ AB, BL $=$ GH, HK ea. to ea.,
and $\angle$ ABL $=$ $\angle$ GHK,
$\therefore$ base AL $=$ base GK:

and

PROP. XXII. CONTINUED.

and ∵ ∠s at E, H, together	$>$	∠ ABC,	
and that ∠ at H	$=$	∠ ABL,	
∴ ∠ at E	$>$	∠ LBC.	
Again, ∵ LB, BC	$=$	DE, EF,	
and that ∠ DEF	$>$	∠ LBC,	
∴ base DF	$>$	base LC :	24. 1.
now GK	$=$	AL,	demon.
∴ DF + GK	$>$	AL + LC ;	
but AL + LC	$>$	AC,	20. 1.
much more ∴ DF, GK	$>$	AC.	

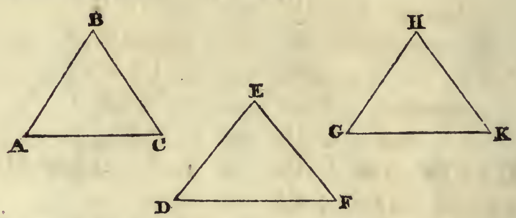
Wherefore every two of these rt. lines AC, DF, GK, are $>$ than the third, and therefore a △ may be made,* the sides of which shall be $=$ AC, DF, GK respectively. * 22. 1.

Q. E. D.

PROP. XXIII.—Problem.

To make a solid angle which shall be contained by three given plane angles, any two of them being greater than the third, and all three together less than four right angles.

Let the three given plane ∠s be ABC, DEF, GHK, of which every two are > than the third, and all of them together < than four rt. ∠s. It is required to make a sol. ∠ contained by three plane ∠s = ABC, DEF, GHK, each to each.



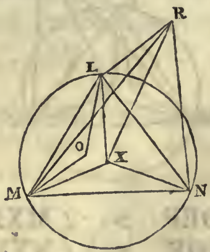
From the rt. lines, which contain the ∠s
cut off AB, BC, DE, EF, GH, HK, all = ea. other;
join AC, DF, GK:
then a △ may be made of three rt. lines = AC, DF, GK; 22.11.
let this △ be LMN, 22.1.
so that AC = LM,
DF = MN,
and GK = LN;
and about △ LMN descr. a ⊙; 4.5.
find X cent. ⊙: 1.3.
which will be either within the △ or on a side, or without it.

First—Let cent X be within the △.
Join LX, MX, NX:
then AB > LX;

or, if

PROP. XXIII. CONTINUED.

or, if AB $\ngtr$ LX,
then it is either $=$ or $<$ LX.

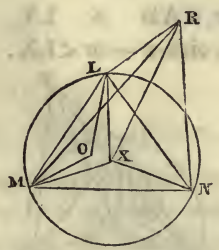


First—Let AB $=$ LX.
Then, $\because$ AB $=$ LX,
and that AB $=$ BC,
and LX $=$ XM,
$\therefore$ the two AB, BC $=$ LX, XM ea. to ea.
and base AC $=$ base LM, constr.
$\therefore \angle$ ABC $= \angle$ LXM :
similarly $\angle$ DEF $= \angle$ MXN,
and $\angle$ GHK $= \angle$ NXL,
$\therefore$ the 3 $\angle$s ABC, DEF, GHK $=$ the 3 $\angle$s LXM, MXN, NXL;
but $\angle$s LXM, MXN, NXL $=$ 4 rt. $\angle$s, 2 cor. 15. 1.
$\therefore \angle$s ABC, DEF, GHK $=$ 4 rt. $\angle$s;
but they are $<$ 4 rt. $\angle$s;
which is absurd.
$\therefore$ AB $\neq$ LX.

Secondly—Let AB $<$ LX.
Then upon LM, and on that side on which is cent. X,
describe a $\triangle$ LOM,
having LO, OM $=$ AB, BC, ea. to ea.
and $\because$ base LM $=$ base AC,
$\therefore \angle$ LOM $= \angle$ ABC. 8. 1.
And AB, i. e. LO $<$ LX, hyp.
$\therefore$ LO, OM fall within the $\triangle$ LXM;
for, if they fell on its sides or without it,
then would LO, OM $=$ or $>$ LX, XM ;
$\therefore \angle$ LOM, i. e. ABC $> \angle$ LXM ; 21. 1.

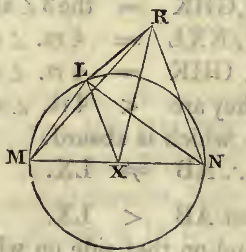
similarly

PROP. XXIII. CONTINUED.



similarly ∠ DEF > ∠ MXN;
and ∠ GHK > ∠ NXL;
∴ ∠s ABC, DEF, GHK > ∠s LXM, MXN, NXL,
i. e. ∠s ABC, DEF, GHK > 4 rt. ∠s;
but ∠s ABC, DEF, GHK < 4 rt. ∠s;
which is absurd.
∴ AB ≮ LX:
and it has been proved that AB ≠ LX;
∴ AB > LX.

SECONDLY—Let cent. X fall on a side MN of the △.

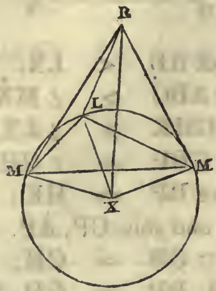


Join XL.
In this case also AB > LX.
For if AB ≯ LX,
it is either AB = or < LX;
let AB = LX;
∴ AB, BC, i. e. DE, EF = MX, XL, i. e. MN:
but MN = DF, constr.
∴ DE, EF = DF;
which is impossible. 20. 1.
∴ AB

PROP. XXIII. CONTINUED.

$\therefore$ AB $\neq$ LX;
neither is AB $<$ LX;
for then a much greater absurdity would follow:
$\therefore$ AB $>$ LX.

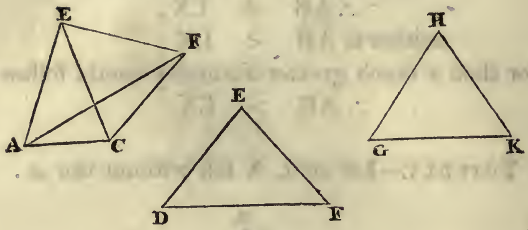
THIRDLY—Let cent. X fall without the $\triangle$.



Join LX, MX, NX.
In this case also AB $>$ LX.
if not,
it is either AB = or < LX.
First—Let AB = LX.
Then as in first case $\angle$ ABC = MXL,
and $\angle$ GHK = $\angle$ LXN;
$\therefore$ whl. $\angle$ MXN = $\angle$s ABC + GHK;
but $\angle$ ABC + GHK $>$ $\angle$ DEF;
$\therefore$ $\angle$ MXN $>$ $\angle$ DEF.
And $\because$ DE, EF = MX, XN ea. to ea.
and that base DF = base MN,
$\therefore$ $\angle$ MXN = $\angle$ DEF; 8. 1.
but also $\angle$ MXN $>$ $\angle$ DEF;
which is absurd.
$\therefore$ AB $\neq$ LX.

Secondly,

PROP. XXIII. CONTINUED.

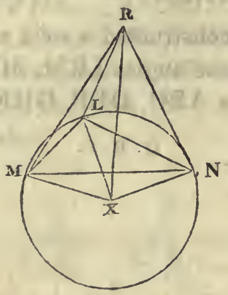


Secondly—Let AB < LX.
Then as in first case ∠ ABC > ∠ MXL.
and ∠ GHK > ∠ LXN.
At B in CB, make ∠ CBP = ∠ GHK;
and make BP = HK;
and join CP, AP.
Then, ∵ CB = GH,
∴ CB, BP = GH, HK ea. to ea.
and they cont. equal ∠ s,
∴ base CP = base GK, i. e. LN;
and, ∵ in the isosceles △ s ABC, MXL,
∠ ABC > ∠ MXL,
∴ ∠ MLX at base > ∠ ACB at base: 32. 1.
similarly, ∵ ∠ GHK or CBP > LXN,
∴ ∠ XLN > ∠ BCP;
∴ whl. ∠ MLN > whl. ∠ ACP.
And ∵ ML, LN = AC, CP ea. to ea.
but that ∠ MLN > ∠ ACP,
∴ base MN > base AP; 24. 1.
but MN = DF.
∴ DF > AP.
Again, ∵ DE, EF = AB, BP ea. to ea.
but that base DF > base AP,
∴ ∠ DEF > ∠ ABP: 25. 1.
but ∠ ABP = ∠ s ABC + CBP, i. e. ABC + GHK.
∴ ∠ DEF

PROP. XXIII. CONTINUED.

$\therefore \angle$ DEF $>$ $\angle$s ABC + GHK ;
but also $\angle$ DEF $<$ $\angle$s ABC + GHK ;
which is impossible :
$\therefore$ AB $\not<$ LX ;
and it has been proved $\neq$ LX ;
$\therefore$ AB $>$ LX.

Now, from X erect XR at rt. $\angle$s to pl. of $\odot$ LMN. 12. 11.



And since it has been demon. in all the cases,
that AB $>$ LX ;
then find a sq. $=$ $AB^2 - LX^2$;
and make RX $=$ to a side of it ;
join RL, RM, RN.
And, $\because$ RX $\perp$ pl. LMN,
$\therefore$ RX $\perp$ LX, MX, NX : 3 def. 11
and $\because$ LX $=$ MX,
and that XR is com. and at rt. $\angle$s to ea.
$\therefore$ base RL $=$ base RM :
similarly RN $=$ RL, or RM ;
$\therefore$ RL, RM, RN $=$ ea. other ;
and, $\because$ XR^2 $=$ $AB^2 - LX^2$,
$\therefore$ AB^2 $=$ $LX^2 + XR^2$:
but RL^2 $=$ $LX^2 + XR^2$, 47. 1.
(for LXR is a rt. $\angle$,)
$\therefore$ AB^2 $=$ RL^2,

and

U

PROP. XXIII. CONTINUED.

and AB	=	RL :	
but ea. of BC, DE, EF, GH, HK	=	AB,	
and ea. of RM, RN	=	RL;	
∴ ea. of AB, BC, DE, EF, GH, HK	=	ea. of RL, RM, RN :	
and ∵ RL, RM	=	AB, BC ea. to ea.	
and that base LM	=	base AC,	
∴ ∠ LRM	=	∠ ABC :	8. 1.
similarly { ∠ MRN	=	∠ DEF,	
and ∠ NRL	=	∠ GHK.	

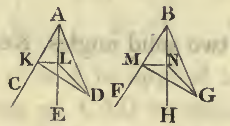
Therefore, there is constructed a solid angle at R, which is contained by three plane angles LRM, MRN, NRL which = the three given pl. ∠ s ABC, DEF, GHK, ea. to ea.

Q. E. F.

PROP. A.—Theorem.

If each of two solid angles be contained by three plane angles, which are equal to one another, each to each; the planes in which the equal angles are, have the same inclination to one another.

Let there be two sol. ∠s at A, B; and let the ∠ at A be contained by the three plane ∠s CAD, CAE, EAD; and the ∠ at B by the three plane ∠s FBG, FBH, HBG; of which the ∠ CAD = the ∠ FBG, and ∠ CAE = ∠ FBH, and ∠ EAD = ∠ HBG; the planes in which the ∠s are, have the same inclination to each other.



In AC take any pt. K;
in pl. CAD, from K, draw KD rt. ∠s to AC;
and in pl. CAE, from K, draw KL also rt. ∠s to AC;
∴ ∠ DKL is the inclination of pl. CAD to pl. CAE. 6 def. 11.
In BF take BM = AK;
and in pls. FBG, FBH, from M, draw MG, MN rt. ∠s to BF;
and ∴ ∠ GMN is the inclin. of pl. FBG to pl. FBH.
Join LD, NG.
Then, ∵ in △ KAD:—∠ KAD = ∠ MBG:—in △ MBG,
and that rt. ∠ AKD = rt. ∠ BMG,
and also the sides adjac. to equal ∠s = ea. other,
viz. AK = MB,
∴ KD = MG, 26. 1.
and AD = BG:
similarly in the △s KAL, MBN,
KL = MN,

PROP. A. CONTINUED.

and AL = BN;
also in the △s LAD, NBG,
LA, AD = NB, BG ea. to ea.
and they contain = ∠s,
∴ base LD = base NG. 4. 1.
Lastly in the △s KLD, MNG,
DK, KL = GM, MN ea. to ea.
and base LD = base NG,
∴ ∠ DKL = ∠ GMN: 8. 1.
but ∠ DKL is the inclin. of pl. CAD to the pl. CAE,
and ∠ GMN is the inclin. of pl. FBG to the pl. FBH,
∴ these pls. have the same inclin. to ea. other.

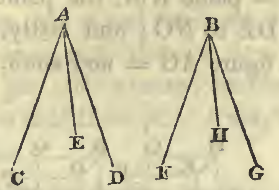
And in the same manner it may be demon. that the other pls. in which the equal ∠s are, have the same inclin. to ea. other.

Therefore, if two solid angles, &c. &c. Q. E. D.

PROP. B.—Theorem.

If two solid angles be contained, each by three plane angles which are equal to one another, each to each, and alike situated; these solid angles are equal to one another.

Let there be two sol. ∠s at A and B, of which the sol. ∠ at A is contained by three plane ∠s, CAD, CAE, EAD; and that at B, by the three plane ∠s FBG, FBH, HBG; of which CAD = FBG; CAE = FBH; and EAD = HBG; then sol. ∠ at A = sol. ∠ at B.



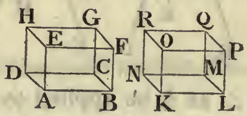
Let the sol. ∠ at A be applied to sol. ∠ at B;
and *first* let the pl. CAD be applied to pl. FBG,
so that pt. A coin. with pt. B;
and that AC coin. with BF:
then, ∵ ∠ CAD = ∠ FBG,
∴ AD coin. with BG:
& ∵ inclin. of pl. CAE to pl. CAD = inclin. of pl. FBH to pl. FBG, A. 11.
and that pl. CAD coin. with pl. FBG,
∴ pl. CAE coin. with pl. FBH:
and ∵ AC coin. with BF,
and that ∠ CAE = ∠ FBH,
∴ AE coin. with BH:
and AD coin. with BG,
∴ pl. EAD coin. with pl. HBG;
∴ sol. ∠ at A coin. with sol. ∠ at B;
and consequently sol. ∠ at A = sol. ∠ at B. 8. ax. 1.

Wherefore, if two solid angles, &c. &c. Q. E. D.

PROP. C.—Theorem.

Solid figures which are contained by the same number of equal and similar planes alike situated, and having none of their solid angles contained by more than three plane angles, are equal and similar to one another.

Let AG, KQ, be two sol. figures contained by the same number of simil. and equal planes, alike situated, viz. let the plane AC be simil. and = plane KM, the plane AF to KP, BG to LQ, GD to QN, DE to NO; and lastly, FH, simil. and = to PR. The sol. figure AG = and simil. to sol. figure KQ.



∵ Sol. ∠ at A is cont. by 3 pl. ∠ s BAD, BAE, EAD,
and sol. ∠ at K is cont. by 3 pl. ∠ s LKN, LKO, OKN,
and that ∠ s BAD, BAE, EAD = ∠ s LKN, LKO, OKN ea.
to ea. hyp.
∴ sol. ∠ at A = sol. ∠ at K : B. 11.
similarly the other sol. ∠ s of the figs. = ea other.
Let sol. fig. AG be applied to sol. fig. KQ ;
and *first*, let pl. fig. AC be applied to pl. fig. KM ;
then rt. line AB coinciding with KL,
the fig. AC cannot but coin. with fig. KM,
(for they are = and simil. ea. other;)
∴ rt. lines AD, DC, CB coin. with KN, NM, ML ea. with ea.
and pts. A, D, C, B coin. with pts. K, N, M, L.
Now sol. ∠ at A coin. with sol. ∠ at K, B. 11.
∴ pl. AF coin. with pl. KP,
(for they are = and simil. ea. other ;)
∴ rt. lines AE, EF, FB coin. with KO, OP, PL,

(and

PROP. C. CONTINUED.

and pts. E, F with pts. O, P.
Similarly fig. AH coin. with fig. KR,
and rt. line DH with NR,
and pt. H with R.
And ∵ sol.∠ at B = sol. ∠ at L,
it may be proved similarly,
that fig. BG coin. with fig. LQ,
and rt. line CG with MQ,
and pt. G with pt. Q.

Then ∵ the pls. and sides of sol. fig. AG coin. with pls. and sides of sol. fig. KQ,

∴ sol. fig. AG = and simil. sol. fig. KQ.

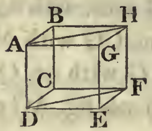
And, in same manner, any other sol. figs. contained by same No. of = and simil. pls. alike situated, and having none of their sol. ∠ s cont. by more than three pl. ∠ s, may be proved to be = and simil. to ea. other.

Wherefore, solid figures, &c. &c. Q. E. D.

PROP. XXIV.—Theorem.

If a solid be contained by six planes, two and two of which are parallel; the opposite planes are similar and equal parallelograms.

Let the sol. DH be cont. by the parall. pls. AC, GF; BG, CE; FB, AE. Its opp. pls. are = and simil. □s.



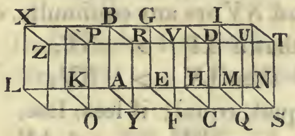
∵ Pl. AC cuts parall. pls. BG, CE,
∴ their secs. AB, CD are ∥ ea. other. 16.11.
Again, ∵ pl. AC cuts parall. pls. BF, AE,
∴ their secs. AD, BC are ∥ ea. other:
and AB ∥ CD,
∴ AC is a □.
In the same way it may be proved,
that ea. of figs. CE, FG, GB, BF, AE is a □.
Join AH, DF,
and, ∵ AB ∥ CD
and BH ∥ CF,
∴ AB, BH which meet ∥ CD, CF which meet:
but they are not in same plane,
∴ ∠ ABH = ∠ DCF: 10.11.
and ∵ AB, BH = DC, CF ea. to ea.
and that ∠ ABH = ∠ DCF,
∴ base AH = base DF; }
and △ ABH = △ DCF; } 4. 1.
now the □ BG = 2 △ ABH, }
also □ CE = 2 △ DCF, } 41. 1.
∴ □ BG = and simil. □ CE.
In the same manner it may be proved,
that □ AC = and simil. □ GF,
and □ AE = and simil. □ BF.

Therefore, if a solid, &c. &c. Q. E. D.

PROP. XXV.—Theorem.

If a solid parallelopiped be cut by a plane parallel to two of its opposite planes; it divides the whole into two solids, the base of one of which shall be to the base of the other, as the one solid is to the other.

Let the sol. ▫ AD be cut by the pl. EV, which is ∥ to opp. pls. AR, HD, and divides the whl. into two sols. AV, ED; then base AF : base FH : : sol. AV : sol. ED.



Produce AH both ways;
and take any No. of rt. lines, HM, MN ea. = EH;
and any No. of rt. lines, AK, KL ea. = EA :
complete the ▫ s, LO, KY, HQ, MS, and sols. LP, KR, HU, MT.

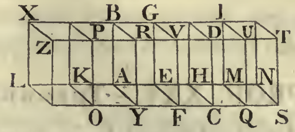
Then, ∵ LK, KA, AE = ea. other,
∴ ▫ s LO, KY, AF = ea. other, } 36. 1.
and ▫ s KX, KB, AG = ea. other, }
and also ▫ s LZ, KP, AR = ea. other, 24.11.
(for they are opp. planes,):

Similarly the ▫ s { EC, HQ, MS = ea. other, } 36. 1.
HG, HI, IN = ea. other, }
& HD, MU, NT = ea. other; 24.11.

∴ 3 pls. of the sol. LP = and simil. 3 pls. of sol. KR,
and also = and simil. 3 pls. of sol. AV:
but the 3 pls. opp. to these 3, = and simil. to them in the several sols. 24.11.
and none of their sol. ∠ s are cont. by more than 3 pl. ∠ s.
∴ the 3 sols. LP, KR, AV = ea. other: C. 11.
similarly 3 sols. ED, HU, MT = ea. other,

∴ sol.

PROP. XXV. CONTINUED.



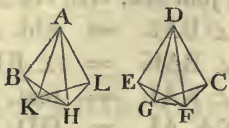
∴ sol. LV is same mult of AV, that base LF is of AF;
and similarly sol. NV is same mult. of ED that base NF is of HF,
and if base LF = base NF,
then sol. LV = sol. NV;
if greater, greater; if less, less.
Now, ∵ there are four mags. viz. bases AF, FH and sols. AV, ED,
and that LF and LV are any equimults. of AF and AV,
and that FN and NV are any equimults. of FH and ED,
and, that if LF > NF,
then LV > NV,
and if equal, equal; if less, less,
∴ base AF : base FH :: sol. AV : sol. ED.

Wherefore, if a solid parallelopiped, &c. &c. Q. E. D.

PROP. XXVI.—Problem.

At a given point in a given right line, to make a solid angle equal to a given solid angle contained by three plane angles.

Let AB be the given rt. line, A the given pt. in it, and D the given solid ∠ contained by the three plane ∠s EDC, EDF, FDC: it is required to make at pt. A in rt. line AB a sol. ∠ = sol. ∠ D.



In DF take any pt. F;
from F, draw FG ⊥ pl. EDC and meeting it in G; 11. 1.
join DG;
at A in AB make ∠ BAL = ∠ EDC; 23. 1.
and in pl. BAL make ∠ BAK = ∠ EDG;
then make AK = DG;
and from K erect KH rt. ∠s to pl. BAL; 12. 11.
and make KH = GF;
join AH:
then sol. ∠ at A = sol. ∠ at D.
Take equal rt. lines AB, DE;
join HB, KB, FE, GE:
and ∵ FG ⊥ pl. EDC,
it makes rt. ∠s with every rt. line meeting it in that pl. 3 def. 11.
∴ ∠s FGD, FGE are rt. ∠s;
similarly ∠s HKA, HKB are rt. ∠s:
and ∵ KA, AB = GD, DE ea. to ea.,
and that these contain = ∠s,
∴ base BK = base EG: 4. 1.
and KH = GF,
also rt. ∠ HKB = rt. ∠ FGE,
∴ HB = FE. 4. 1.

Again,

PROP. XXVI. CONTINUED.

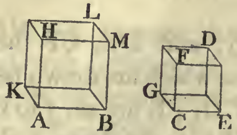
Again, ∵ AK, KH = DG, GF ea. to ea.,
and contain rt. ∠s,
∴ base AH = base DF;
and AB = DE,
∴ HA, AB = FD, DE ea. to ea.;
and base HB = base FE,
∴ ∠ BAH = ∠ EDF. 8. 1.
Similarly ∠ HAL = ∠ FDC:
for make AL = DC;
and join KL, HL, GC, FC.
Then, ∵ whl. ∠ BAL = whl. ∠ EDC, } constr.
and that ∠ BAK = ∠ EDG, }
∴ rem. ∠ KAL = rem. ∠ GDC.
And ∵ KA, AL = GD, DC ea. to ea.,
and contain equal ∠s,
∴ base KL = base GC; 4. 1.
and KH = GF,
∴ LK, KH = CG, GF ea. to ea.,
and they cont. rt. ∠s,
∴ base HL = base FC.
Again, ∵ HA, AL = FD, DC ea. to ea.,
and that base HL = base FC,
∴ ∠ HAL = ∠ FDC. 8. 1.
Now, ∵ 3 pl. ∠s BAL, BAH, HAL, which contain sol. ∠ at A, } = { 3 pl. ∠s EDC, EDF, FDC which contain sol. ∠ at D } ea. to ea.,
and that they are situated in same order,
∴ sol. ∠ at A = sol. ∠ at D. B. 11.

Therefore at a given point in a given rt. line, a solid angle has been made equal to a given solid angle contained by three plane angles. Q. E. F.

PROP. XXVII.—Problem.

To describe from a given right line a solid parallelopiped similar and similarly situated to one given.

Let AB be the given rt. line, and CD the given Sol. □. It is required to describe from AB a Sol. □ simil. and similarly situated to Sol. □ CD.



At A in AB make a sol. ∠ = sol. ∠ at C; 26. 11.

and let the three pl. ∠s BAK, KAH, HAB contain it;

so that { ∠ BAK = ∠ ECG,
∠ KAH = ∠ GCF,
∠ HAB = ∠ FCE;

and make EC : CG :: BA : AK,
and GC : CF :: KA : AH; } 12. 6.

∴ ex æquali. EC : CF :: BA : AH. 22. 6.

Complete the □ BH and sol. AL.

and ∵ EC : CG :: BA : AK,

then the sides about equal ∠s ECG, BAK are propors.;

∴ □ BK simil. □ EG:

similarly □ KH simil. □ GF,

and □ HB simil. □ FE,

∴ 3 □s of the sol. AL simil. 3 □s of sol. CD;

and ∴ the three opp. ones in ea. sol. = and simil. to these ea. to ea. 24. 11.

Also, ∵ the pl. ∠s which contain the sol. ∠s of the figs. } = ea. to ea.,

and that they are situated in same order,

∴ the sol. ∠s = ea. to ea.; B. 11.

∴ sol. AL simil. sol. CD. 11 def. 11.

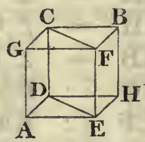
Wherefore from a given rt. line AB a Sol. □ AL has been descr. simil. and similarly situated to the given Sol. □ CD.

Q. E. F.

PROP. XXVIII.—Theorem.

If a solid parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes: it shall be cut into two equal parts.

Let AB be a Sol. ▭, and DE, CF the diags. of the opp. ▭ s AH, GB, viz. those which are drawn between the equal ∠ s in ea. And because CD, FE are ea. ∥ to GA, and not in same pl. with it, CD is ∥ FE* ∴ the diags. CF, DE are in *9. 11.
the pl. in which the ∥ s are, and are themselves ∥ : † †16. 11.
and the pl. DF shall cut the sol. AB into two = parts.



∵ △ GCF = △ CBF, 34. 1.
and △ DAE = △ DHE,
and that ▭ CA = and simil. opp. ▭ BE, 24. 11.
and ▭ GE = and simil. opp. ▭ CH,

∴ the PRISM cont. by △ s CGF, DAE and the 3 ▭ s CA, GE, EC } = { the PRISM cont. by △ s CBF, DHE and the 3 ▭ s BE, CH, EC;

for they are contained by the same No. of equal and similar pls. alike situat. and none of their sol. ∠ s are cont. by more than three pl. ∠ s, C. 11.

∴ solid AB is cut into two = parts by pl. DF.

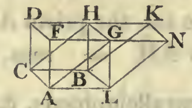
Q. E. D.

N.B. "The insisting right lines of a parallelopiped, mentioned in the next and some following propositions, are the sides of the parallelograms between the base and the opposite plane parallel to the base."

PROP. XXIX.—THEOREM.

Solid parallelopipeds upon the same base, and of the same altitude, the insisting right lines of which, are terminated in the same right lines in the plane opposite to the base, are equal to each other.

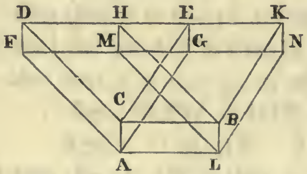
Let the Sol. □ s AH, AK be upon same base AB, and of the same altitude, and their insisting rt. lines AF, AG, LM, LN; CD, CE, BH, BK be terminated in same rt. lines FN, DK. Then the solid AH = solid AK.

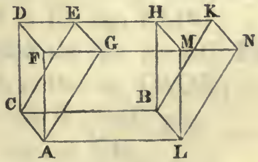


FIRST—Let □ s DG, HN, opp. to base AB, have a com. side HG.

And, ∵ sol. AH is cut by a pl. CG passing thro diags. AG, CH,
∴ sol. AH is cut into two = parts by pl. CG; 28. 11.
∴ sol. AH = 2 prism between △ s ALG, CBH;
similarly sol. AK is cut into two = parts by pl. BG;
and ∴ sol. AK = 2 of same prism ALG, CBH;
∴ sol. AH = sol. AK.

SECONDLY—Let □ s DM, EN opp. to base AB have no com. side.





Then, ∵ □ CH = □ CK,
∴ CB = ea. of DH, EK, 34. 1.
and ∴ DH = EK;

add

PROP. XXIX. CONTINUED.

add or take away com. part HE,

then DE = HK;

∴ also △ CDE = △ BHK; 38. 1.

and □ DG = □ HN: 36. 1.

similarly △ AFG = △ LMN;

also □ CF = □ BM; } 24. 11.
and □ CG = □ BN, }

(for they are opp.);

∴ the PRISM cont. by the △ s AFG, CDE and □ s AD, DG, GC } = { the PRISM cont. by the △ s LMN, BHK & □ s BM, MK, KL; } C. 11.

Then, if prism LMNBHK be taken from the sol. whose base is the □ AB and DN the □ opp. to it,

and, if from the same sol. the prism AFGCDE, be taken,

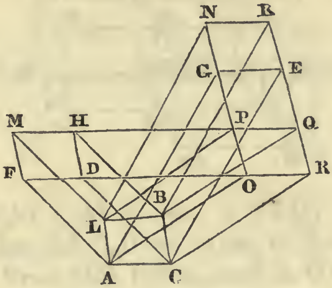
∴ rem. sol. □ AH = rem. sol. □ AK.

Therefore solid parallelopipeds, &c. &c. Q. E. D.

PROP. XXX.—Theorem.

Solid parallelopipeds upon the same base, and of the same altitude, the insisting right lines of which are not terminated in the same right lines in the plane opposite to the base, are equal to each other.

Let the Sol. ▫s CM, CN be upon same base AB, and of the same altitude, but their insisting rt. lines AF, AG, LM, LN, CD, CE, BH, BK not term. in same rt. lines. Then the sol. CM = sol. CN.



Prod. FD, MH, and NG, KE;
and let them meet in the pts. O, P, Q, R:
join AO, LP, BQ, CR.
Then, ∵ pl. LH ∥ opp. pl. AD,
and that the pl. LH is that in which are the ∥ s LB, MQ,
also that it is the pl. in which is the fig. BLPQ;
and that the pl. AD is that in which are the ∥ s AC, FR,
also that it is the pl. in which is the fig. CAOR;
∴ figs. BLPQ, CAOR are are in parall. pls.
Again, ∵ pl. AN ∥ opp. pl. CK,
and that the pl. AN is that in which are the ∥ s AL, ON,
also that it is the pl. in which is the fig. ALPO;
and that the pl. CK is that in which are the ∥ s CB, RK,

 also

PROP. XXX. CONTINUED.

also that it is the pl. in which is the fig. CBQR;
∴ figs. ALPO, CBQR are in parall. pls.
Now pls. ACBL, ORQP ∥ ea. other,
∴ fig. CP is a sol. □:
but sol. CM = sol. CP, 29.11.
(for they are on the same base AB and their insist. rt. lines are term. in same rt. lines FR, MQ,)
and sol. CP = sol. CN, 29.11.
(for they are on same base AB and their insist. rt. lines are term. in same rt. lines, ON, RK.)
∴ sol. CM = sol. CN.

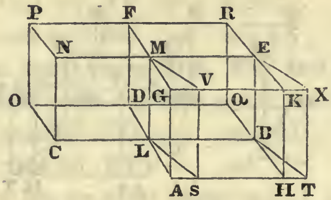
Wherefore solid parallelopipeds, &c. &c. Q. E. D.

PROP. XXXI.—THEOREM.

Solid parallelopipeds, which are upon equal bases, and of the same altitude, are equal to each other.

Let the Sol. ▫ s AE, CF be upon equal bases AB, CD, and of the same altitude; then sol. AE = sol. CF.

FIRST—Let the insisting rt. lines be at rt. ∠ s to the bases AB, CD, and let the bases be placed in the same pl. and so as that sides CL, LB be in one rt. line; therefore rt. line LM which is right ∠ s to the pl. in which the bases are, in pt. L, shall be com.* to the two sols. AE, CF; let the other insist. lines be AG, HK, BE; DF, OP, CN. * 13. 11.



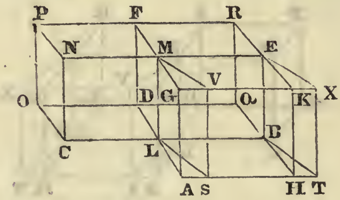
And *first* let ∠ ALB = ∠ CLD;
then AL, LD are in one rt. line. 14. 1.
Prod. OD, HB to meet in Q;
and complete the Sol. ▫ LR, whose base is ▫ LQ
and LM one of its insist. rt. lines.
Now, ∵ ▫ AB = ▫ CD,
∴ base AB : base LQ : : base CD : base LQ : 7. 5.
And ∵ Sol. ▫ AR is cut by pl. LE,
and that pl. LE ∥ opp. pls. AK, DR,
∴ base AB : base LQ : : sol. AE : sol. LR. 25. 11.
Again, ∵ Sol. ▫ CR is cut by pl. LF,
and that pl. LF ∥ opp. pls. CP, BR,
∴ base CD : base LQ : : sol. CF : sol. LR.
Now it was proved
that base AB : base LQ : : base CD : base LQ,
and ∴ sol. AE : sol. LR : : sol. CF : sol. LR;
∴ sol. AE = sol. CF. 9. 5.

 Secondly,

PROP. XXXI. CONTINUED.

Secondly—Let Sol. ▭s SE, CF be upon = bases SB, CD, and of same altitude, and again let their insist. rt. lines be rt. ∠s to their bases; and place the bases SB, CD in same pl. so that CL, LB be in a rt. line.

But let ∠ SLB ≠ CLD;
then shall sol. SE = sol. CF.
Prod. DL, TS to meet in A,
from B, draw BH ∥ DA;
and let HB, OD prod. meet in Q,
complete sols. AE, LR;
∴ sol. AE = sol. SE, 29.11.
(for they are on same base LE, and of same alt. and their insist. rt. lines are term. in same rt. lines AT, GX.)



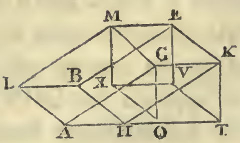
And ∵ ▭ AB = ▭ SB, 35. 1.
(being on same base LB, and between same ∥s LB, AT,)
and that base SB = base CD,
∴ AB = CD:
and ∠ ALB = ∠ CLD,
∴ by 1st case sol. AE = sol. CF;
but sol. AE = sol. SE, demon.
∴ sol. SE = sol. CF.

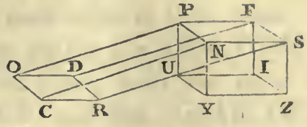
Secondly—Let the insist. rt. lines be not rt. ∠s to bases AB, CD.

From the pts. G, K, E, M; N, S, F, P,
draw { GQ, KT, EV, MX; NY, SZ, FI, PU, } ⊥ pls. of the bases AB, CD; 11.11.
and let them meet these pls. in Q, T, V, X; Y, Z, I, U;
and join QT, TV, VX, XQ; YZ, ZI, IU, UY.
Now, ∵ GQ, KT are rt. ∠s to the same pl.
∴ GQ, KT ∥ ea. other: 6.11.
and MG, EK ∥ ea. other:

and,

PROP. XXXI. CONTINUED.





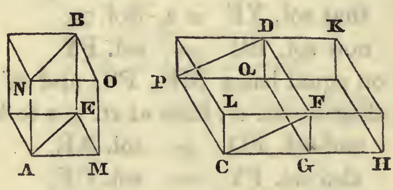
and, ∵ MG, GQ ‖ EK, KT,
but are not in same pl.,
and that pl. MQ passes thro. MG, GQ,
and pl. ET passes thro. EK, KT,
∴ pl. MQ ‖ pl. ET : 15. 11.
similarly pl. MV ‖ pl. GT.
∴ sol. QE is a Sol. ▫ .
In the same manner it may be proved,
that sol. YF is a Sol. ▫ ;
now sol. EQ = sol. FY,
(for they are on equal bases MK, PS, and of same alt. and
have their insist. rt. lines at rt. ∠ s to bases,)
and sol. EQ = sol. AE, 29 or 30. 11.
also sol. FY = sol. CF,
(for they are on same bases and of same alt.)
∴ sol. AE = sol. CF.

Wherefore solid parallelopipeds, &c. &c. Q. E. D.

PROP. XXXII.—Theorem.

Solid parallelopipeds which have the same altitude, are to each other as their bases.

Let AB, CD be Sol. □ s of same altitude. They shall be to ea. other as their bases; i. e. base AE : base CF : : sol. AB : sol. CD.



To rt. line FG, apply a □ FH = □ AE, cor. 45. 1.
so that, ∠ FGH = ∠ LCG:
complete Sol. □ GK, on base FH,
and having FD one of its insisting rt. lines;
∴ Sols. GK, AB are of same alti.
and ∴ sol. AB = sol. GK. 31. 11.
And ∵ the Sol. □ CK is cut by pl. DG,
and that pl. DG ∥ opp. pls.,
∴ base HF : base FC : : sol. GK : sol. DC: 25. 11.
but base HF = base AE,
and sol. GK = sol. AB,
∴ base AE : base FC : : sol. AB : sol. CD.

Wherefore solid parallelopipeds, &c. &c. Q. E. D.

Cor.

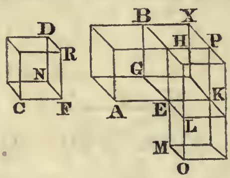
Cor. From this it is manifest, that prisms upon triangular bases, of same altitude, are to each other as their bases.

Let the prisms whose bases are the △s AEM, CFG, and NBO, PDQ the △s opp. to the bases, have the same altitude; and complete ▱s AE, CF, and Sol. ▱s AB, CD, in the first of which let MO be one of the insist. rt. lines, and GQ in the other. And ∵ Sol. ▱s AB, CD have same alt. they shall be to ea. other as base AE : base CF; ∴ the prisms which are the halves,* shall be to each other as the base AE : base CF, i. e. as △ AEM : △ CFG. *28. 11.

PROP. XXXIII.—Theorem.

Similar solid parallelopipeds are to each other in the triplicate ratio of their homologous sides.

Let AB, CD be similar Sol. □s, and the side AE homol. to the side CF. The solid AB shall have to the sol. CD, the triplicate ratio of that which AE has to CF, viz. AB : CD : : tripl. of AE : CF.



Produce AE, GE, HE;
in these produced,
take { EK = CF,
EL = FN,
and EM = FR:
and complete □ KL and the sol. KO.
Then, ∵ sol. AB simil. sol. CD,
∴ ∠ AEG = ∠ CFN;
and ∴ ∠ KEL = ∠ CFN: 15. 1.
and, ∵ KE, EL = CF, FN ea. to ea.
and that ∠ KEL = ∠ CFN,
∴ □ KL = and simil. □ CN:
similarly { □ MK = and simil. □ CR,
and □ OE = and simil. □ FD;
∴ three □s of sol. KO = and simil. three □s of sol. CD,
and the three opp. □s in ea. sol. are = and simil. to these; 24. 11.
∴ sol. KO = and simil. sol. CD. C. 11.
Complete □ GK and the sols. EX, LP on bases GK, KL,
so that EH be an insist. rt. line com. to ea. of them;
and consequently they are of same alt. with sol. AB.

Again,

PROP. XXXIII. CONTINUED.

Again, ∵ sol. AB simil. sol. CD,
and permut. AE : CF : : EG : FN : : EH : FR,

and that { FC = EK,
FN = EL,
FR = EM,

∴ AE : EK : : EG : EL : : EH : EM;
but AE : EK : : □ AG : □ GK, } 1. 6.
and GE : EL : : □ GK : □ KL, }
also HE : EM : : □ PE : □ KM, 1. 6.
∴ □ AG : □ GK : : □ GK : □ KL : : □ PE : □ KM;
but □ AG : □ GK : : sol. AB : sol. EX, }
and □ GK : □ KL : : sol. EX : sol. PL, } 25. 11.
and □ PE : □ KM : : sol. PL : sol. KO, }
∴ sol. AB : sol. EX : : sol. EX : sol. PL : : sol. PL : sol. KO;
∴ sol. AB : sol. KO : : tripl. of sol. AB : sol. EX; 11 def. 5.
but AB : EX : : □ AG : □ GK : : rt. line AE : rt. line EK,
∴ sol. AB : sol. KO : : tripl. of AE : EK:
now sol. KO = sol. CD,
and EK = EF,
∴ sol. AB : sol. CD : : tripl. of AE : CF.

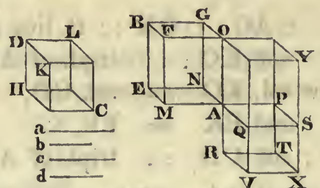
Wherefore similar solid parallelopipeds, &c. &c. Q. E. D.

Cor. From this it is manifest, that, if four right lines be continual proportionals, as the first is to the fourth, so is the solid parallelopiped described from the first to the similar solid similarly described from the second; because the first right line has to the fourth the triplicate ratio of that which it has to the second.

PROP. D.—Theorem.

Solid parallelopipeds contained by parallelograms equiangular to each other, each to each, that is, of which the solid angles are equal, each to each, have to each other the ratio which is the same with the ratio compounded of the ratios of their sides.

Let AB, CD be Sol. □ s, of which AB is contained by the □ s AE, AF, AG which are equiang. ea. to ea. to □ s CH, CK, CL which contain the sol. CD. Then the ratio of sol. AB : sol. CD shall be the same with that which is compounded of the ratios of the sides AM : DL, AN : DK, and AO : DH which is the same as AM : DH.* * def. A. 5.



Prod. MA, NA, OA to P, Q, R,

so that { AP = DL,
AQ = DK,
and AR = DH :

and complete the Sol. □ AX

contd. by □ s AS, AT, AV = and simil. □ s CH, CK, CL ea. to ea.;

∴ sol. AX = sol. CD : C. 11.

also complete sol. AY whose base is AS, and AO an insist. line.

Take any rt. line a :

and make $a : b$: : MA : AP,

and $b : c$: : NA : AQ,

and $c : d$: : OA : AR.

Now, ∵ □ AE is equiang. to □ AS,

∴ AE : AS : : $a : c$; 23. 6.

and ∵ sols. AB, AY are between parall. pls. BOY, EAS,

they

PROP. D. CONTINUED.

they are of the same altitude,
∴ sol. AB : sol. AY :: base AE : base AS, i.e. :: $a : c$; 32. 11.
and AY : AX :: base OQ : base QR, i.e. :: OA : AR,
i.e. :: $c : d$:

now ∵ sol. AB : sol. AY :: $a : c$,
and that sol. AY : sol. AX :: $c : d$,
∴ ex æquo AB : AX :: $a : d$;
but CD = AX,
∴ AB : CD :: $a : d$:
but $a : d$ is comp. of $a : b$, $b : c$, and $c : d$, def. A. 5.
which also is the same with MA : AP, NA : AQ, and OA : AR ea. to ea.,
and sides AP, AQ, AR = sides DL, DK, DH ea. to ea.,
∴ sol. AB : sol. CD :: AM : AH;
i.e. sol. AB : sol. CD is same with the ratio which is compounded of the ratios of their sides AM : DL, AN : DK, and AO : DH.

Wherefore solid parallelopipeds, &c. &c. Q. E. D.

PROP. XXXIV.—Theorem.

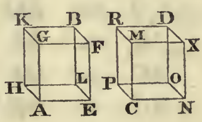
The bases and altitudes of equal solid parallelopipeds, are reciprocally proportional: and if the bases and altitudes be reciprocally proportional, the solid parallelopipeds are equal.

If the Sol. □s AB, CD be equal to ea. other; then shall their bases and alts. be reciprocally propor.

And if the bases and alts. of the Sol. □s AB, CD be recip. propor. Then shall sol. AB = sol. CD.

First case—Let insist. rt. lines AG, EF, LB, HK; CM, NX, OD, PR be rt. ∠s the bases.

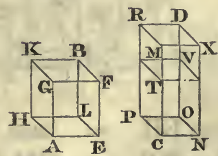
First—Let AB, CD be equal Sol. □s; their bases shall be reciprocally proportional to their altitudes; i. e. base EH : base NP : : CM : AG.



First—Let base EH = NP.
then ∵ sol. AB = sol. CD,
∴ CM = AG;
for if EH = NP,
but alti. CM ≠ alti. AG,
then sol. AB ≠ sol. CD;
but by hyp. sol. AB = sol. CD,
∴ alti. CM is not ≠ alti. AG;
i. e. CM = AG;
∴ base EH : base NP : : CM : AG.

Secondly

PROP. XXXIV. CONTINUED.

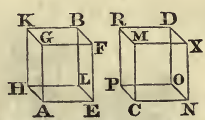


Secondly—Let base EH $\neq$ base NP,
but let EH $>$ NP.
Now, $\because$ sol. AB $=$ sol. CD,
then CM $>$ AG;
for if CM $\ngtr$ AG,
then in this case also sol. AB $\neq$ sol. CD;
but sol. AB $=$ sol. CD,
$\therefore$ CM $>$ AG.
Then make CT $=$ AG,
and complete sol. □ CV whose base is NP and alt. CT.
Now, $\because$ sol. AB $=$ sol. CD,
$\therefore$ AB : CV :: CD : CV; 7. 5.
but AB : CV :: base EH : base NP, 32. 11.
(for sols. AB, CV are same alt.)
also CD : CV :: base MP : PT :: rt. line MC : CT ::
CT : AG, 25. 11. and 1. 6.
$\therefore$ base EH : base NP :: MC : AG.

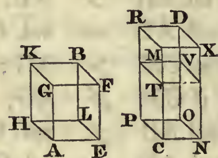
Secondly—Let the bases of sol. □s AB, CD be reciprocally proportional to their alt. i. e. EH : NP :: CM : AG. Then shall sol. AB $=$ sol. CD.

First

PROP. XXXIV. CONTINUED.



First—Let base EH = base NP.
Then, $\because$ EH : NP : : CM : AG,
$\therefore$ alt CM = alt. AG, A. 5.
and conseq. sol. AB = sol. CD. 31.11.



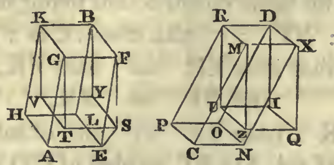
Secondly—Let base EH $\neq$ base NP,
and let EH $>$ NP.
Then $\because$ EH : NP : : CM : AG,
$\therefore$ CM $>$ AG. A. 5.
Take CT = AG,
and complete, as before, sol. CV,
and $\because$ EH : NP : : CM : AG,
and that AG = CT,
$\therefore$ EH : NP : : MC : CT;
but base EH : base NP : : sol. AB : sol. CV, 32.11.
(for sols. AB, CV have same alt.)
and MC : CT : : base MP : base PT : : sol.CD : sol. CV, 25.11.
$\therefore$ sol. AB : sol. CV : : sol. CD : sol. CV,
$\therefore$ sol. AB = sol. CD. 9. 5.

Second case—Let the insist. rt. lines FE, BL, GA, KH; XN, DO, MC, RP not be at rt. $\angle$s to bases of the solids: and from pts. F, B, K, G; X, D, R, M draw $\perp$s to the pls. in which are the bases EH, NP, meeting these pls. in the

PROP. XXXIV. CONTINUED.

the pls. S, Y, V, T; Q, I, U, Z; and complete the sols. FV, XU, which shall be Sol. □ s, (31. 11.)

FIRST—Let the sols. AB, CD be equal, and in this case also, their bases shall be reciprocally proportional to their altitudes, i. e. EH : NP : : alti. of sol. CD : alti. of sol. AB.

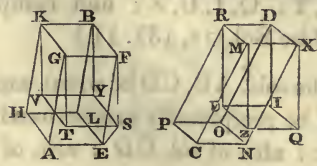


∵ Sol. AB = sol. CD,
and that sol. BT = sol. BA, 29 or 30. 11.
(for they are on same base FK, and of same alt.)
also that sol. DC = sol. DZ; 29 or 30. 11.
(for they are on same base XR, and of same alt.)
∴ sol. BT = sol. DZ;
but of equal Sol. □ s, whose insist. rt. lines are at rt. ∠ s to their bases, the bases are reciprocally propor. to the altitudes; as was proved in the *first case;*
∴ base FK : base XR : : alti. of sol. DZ : alti. of sol. BT.
Now FK = base EH,
and XR = base NP,
∴ base EH : base NP : : alti. of sol. DZ : alti. of sol. BT;
but alts. of sols. DZ, DC as also of sols. BT, BA, are the same,
∴ base EH : base NP : : alti. of sol. DC : alti. of sol. BA;
i. e. the bases of the Sol. □ s AB, CD are reciprocally proportional to their altitudes.

SECONDLY—Let the bases of the Sol. □ s AB, CD be recip. propor. to their alts. viz. EH : NP : : alti. of CD : to alti. of sol. AB; then shall sol. AB = sol. CD.

The

PROP. XXXIV. CONTINUED.



The same construction being made,

∵ EH : NP :: alti. of sol. CD : alti. of AB,

and that base EH = base FK,

and NP = XR,

∴ base FK : base XR :: alti. of sol. CD : alti. of AB;

now alts. of sol. AB, BT, as also of CD, DZ are same,

∴ base FK : base XR :: alti. of DZ : alti. of BT;

i. e. bases of the sols. BT, DZ are recip. propor. to alts.

and their insist. rt. lines are rt. ∠s to the bases;

∴ as before proved, sol. BT = sol. DZ;

but sol. BT = sol. BA,

and DZ = DC,

(for they are on same bases and of same alt.)

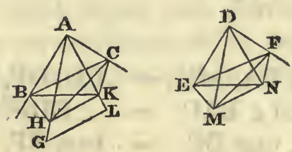
∴ solid AB = solid CD.

Q. E. D.

PROP. XXXV.—Theorem.

If, from the vertices of two equal plane angles, there be drawn two right lines elevated above the planes in which the angles are, and containing equal angles with the sides of those angles, each to each; and if in the lines above the planes there be taken any points, and from them perpendiculars be drawn to the planes in which the first named angles are; and from the points in which they meet the planes, right lines be drawn to the vertices of the angles first named: these right lines shall contain equal angles with the right lines which are above the planes of the angles.

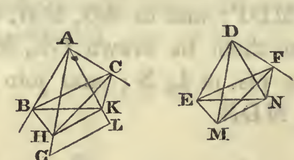
Let BAC, EDF be two equal pl. ∠s; and from pts. A, D let AG, DM be elevated above the pls. of the ∠s, making equal ∠s with their sides, ea. to ea. viz. ∠ GAB = ∠ MDE, and ∠ GAC = ∠ MDF; and in AG, DM, let any pts. G, M be taken, and from them be drawn GL, MN ⊥ pls. BAC, EDF meeting those pls. in L, N; and join LA, ND. Then shall ∠ GAL = ∠ MDN.



Make AH = DM;
and thro. H, draw HK ∥ GL;
but GL ⊥ pl. BAC,
∴ HK ⊥ pl. BAC;
from K, N draw { KB, KC, / NE, NF, } ⊥ { AB, AC, / DE, DF;
and join HB, BC, ME, EF.

PROP. XXXV. CONTINUED.

∵ HK ⊥ pl. BAC,
and ∵ pl. HBK passes thro. HK,
∴ pl. HBK is rt. ∠ s to pl. BAC ; 18.11.
and AB is drawn, in pl. BAC, rt. ∠ s to com. sec. BK of the two pls.
∴ AB ⊥ pl. HBK ; 4 def. 11.
and ∵ BH meets AB in pl. HBK,
∴ ABH is a rt. ∠ ; 3 def. 11.
similarly DEM is a rt. ∠,
and ∴ ∠ DEM = ∠ ABH ;
and ∠ HAB = ∠ MDE ;
∴ in the two △ s HAB, MDE,
two ∠ s of one = two ∠ s of the other, ea. to ea.
also the sides opp. to equal ∠ s = ea. other,
viz. AH = DM,
and ∴ AB = DE. 26. 1.
In the same manner, if HC, MF be joined, it may be demon.
that AC = DF :



∴ BA, AC = ED, DF, ea. to ea.
and ∠ BAC = ∠ EDF,
∴ base BC = base EF, } 4. 1.
and ∠ ABC = ∠ DEF ; }
and rt. ∠ ABK = rt. ∠ DEN,
∴ rem. ∠ CBK = rem. ∠ FEN :
similarly ∠ BCK = ∠ EFN :
∴ in the two △ s BCK, EFN,
two ∠ s of the one = two ∠ s of the other, ea. to ea.
also sides adjac. to equal ∠ s = ea. other,
viz. BC = EF,
∴ BK = EN

also

PROP. XXXV. CONTINUED.

also AB	=	DE,	
∴ AB, BK	=	DE, EN, ea. to ea.	
and these contain rt. ∠ s,			
∴ base AK	=	base DN :	
and ∵ AH	=	DM,	
∴ AH^2	=	DM^2;	
but $AK^2 + KH^2$	=	AH^2,	47. 1.
(for AKH	is a	rt. ∠ ,)	
and $DN^2 + NM^2$	=	DM^2,	
(for DNM	is a	rt. ∠ ,)	
∴ $AK^2 + KH^2$	=	$DN^2 + NM^2$;	
and of these, AK^2	=	DN^2,	
∴ rem. KH^2	=	rem. NM^2;	
and ∴ KH	=	NM :	
now ∵ HA, AK	=	MD, DN ea. to ea.	demon.
and that base HK	=	base MN,	
∴ ∠ HAK	=	∠ MDN.	8. 1.

Q. E. D.

Cor. From this it is manifest, that if from the vertices of two equal plane angles, there be elevated two equal right lines containing equal angles with the sides of the angles, each to each ; the perpendiculars drawn from the extremities of the equal right lines to the planes of the first angles are equal to each other.

Another demonstration of the corollary.

Let the pl. ∠ s BAC, EDF = ea. other, and let AH, DM be two equal rt. lines elevated above the pls. of the ∠ s, containing equal ∠ s with BA, AC, ED, DF ea. to ea. viz. ∠ HAB = ∠ MED, and ∠ HAC = ∠ MDF ; and from H, M let HK, MN be ⊥ s to pls. BAC, EDF ; then shall HK = MN.

∵ sol. ∠ at A is cont. by three pl. ∠ s BAC, BAH, HAC, and sol. ∠ at D is cont. by three pl. ∠ s EDF, EDM, MDF,

PROP. XXXV. CONTINUED.

& that ∠s BAC, BAH, HAC = ∠s EDF, EDM, MDF, ea. to ea.

∴ sol. ∠ at A = sol. ∠ at D :

and ∴ also sol. ∠ at A coin. with sol. ∠ at D;

for, if pl. ∠ BAC be applied to pl. ∠ EDF,

then AH shall coin. with DM; B. 11.

and ∵ AH = DM,

∴ pt. H coin. with M;

∴ HK which is ⊥ to pl. BAC, shall coin. with MN ⊥ pl. EDF, 13. 11.

(for these pls. coin. with ea. other.)

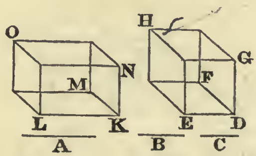
∴ HK = MN.

Q. E. D.

PROP. XXXVI.—Theorem.

If three right lines be proportionals, the solid parallelopiped described from all three as its sides, is equal to the equilateral parallelopiped described from the mean proportional; one of the solid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the solid angles of the other figure.

Let A, B, C, be three proportionals, viz. A : B :: B : C. The sol. described from A, B, C shall be = to the equilat. sol. described from B, equiang. to the other.



Take a sol. ∠ D cont. by 3 pl. ∠ s EDF, FDG, GDE;
make ED, DF, DG ea. = B;
and complete the Sol. ▭ DH.
Make LK = A;
at K in LK,
make a sol. ∠ cont. by 3 pl. ∠ s LKM, MKN, NKL, 26.11.
so that these three pl. ∠ s = ∠ s EDF, FDG, GDE ea. to ea.

make KN = B;
and KM = C;
and complete the Sol. ▭ KO.
Then ∵ A : B :: B : C,
and that A = LK,
and B = DE or DF,
and C = KM,
∴ LK : ED :: DF : KM;
i. e. the sides about equal ∠ s are recip. propor.

∴

PROP. XXXVI. CONTINUED.

∴ □ LM = □ EF; 14. 6.

now since pl. ∠ EDF = pl. ∠ LKM,

and the two equal rt. lines DG, KN are drawn from their verts. above the pls.

and that these cont. equal ∠s with their sides,

∴ the ⊥s from G, N to the pls. EDF, LKM = ea. other; cor.35.11.

∴ sols. KO, DH are of same alt.

Also base LM = base EF,

∴ sol. KO = sol. DH; 31.11.

now sol. KO is descr. from the three rt. lines, A, B, C;

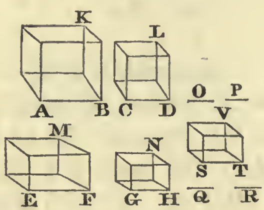
and sol. DH is descr. from B.

Therefore, if three rt. lines, &c. &c. Q. E. D.

PROP. XXXVII.—Theorem.

If four right lines be proportionals, the similar solid parallelopipeds similarly described from them shall also be proportionals. And if the similar parallelopipeds similarly described from four right lines be proportionals, the right lines shall be proportionals.

First—Let the four rt. lines AB, CD, EF, GH be proportionals, viz. AB : CD : : EF : GH; and let the similar Sol. □s AK, CL, EM, GN be similarly described from them. Then shall AK : CL : : EM : GN.



Make AB, CD, O, P continued propors., as also EF, GH, Q, R. } 11. 6.

And ∵ AB : CD : : EF : GH,

then is CD : O : : GH : Q, and O : P : : Q : R, } 11. 5.

∴ ex æquali AB : P : : EF : R; 22. 5.

but AB : P : : sol. AK : sol. CL, and EF : R : : sol. EM : sol. GN, } cor. 33. 11.

∴ sol. AK : sol. CL : : sol. EM : sol. GN. 11. 5.

Secondly—Let sol. AK : sol. CL : : sol. EM : sol. GN.

Then shall AB : CD : : EF : GH.

make AB : CD : : EF : ST;

and from ST descr. a Sol. □ SV similar and similarly situated to sol. EM or GN.

and ∵ AB : CD : : EF : ST,

and

PROP. XXXVII. CONTINUED.

and that from AB, CD, are similarly descr. Sol. □s AK, CL,
and also from EF, ST, are similarly descr. Sol. □s EM, SV,

∴ AK : CL :: EM : SV;
but AK : CL :: EM : GN, hyp.
∴ GN = SV; 9. 5.

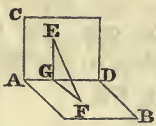
but also GN is similar and similarly descr. to SV,
∴ pls. which cont. sols. GN, SV are similar and = ea. other;
and homol. side GH = homol. side ST.
And ∵ AB : CD :: EF : ST,
and that ST = GH,
∴ AB : CD :: EF : GH.

Therefore if four right lines, &c. &c. Q. E. D.

PROP. XXXVIII.—Theorem.

"If a plane be perpendicular to another plane, and a "right line be drawn from a point in one of the planes "perpendicular to the other plane, this right line shall fall "on the common section of the planes." *

"Let pl. CD be ⊥ pl. AB, and AD their sec. and let any pt. E be taken in the pl. CD: then the ⊥ drawn from E to the pl. AB shall fall on AD.



For if it does not,
let it, if possible, fall off it, as EF;
and let EF meet pl. AB in F;
and from F in pl. AB draw FG ⊥ AD, 12. 1.
and then also is FG ⊥ pl. CD; 4 def. 11.
join EG;
now ∵ FG ⊥ pl. CD,
and that EG meets FG in pl. CD,
∴ FGE is a rt. ∠; 3 def. 11.
but also EF ⊥ pl. AB,
∴ EFG is a rt. ∠;
∴ two of the ∠s of △ EFG = 2 rt. ∠s;
which is absurd.
∴ The perpendicular from E to pl. AB does not fall off AD,
∴ the perpendicular from E to pl. AB falls on AD.

Therefore if a plane," &c. &c. Q. E. D.

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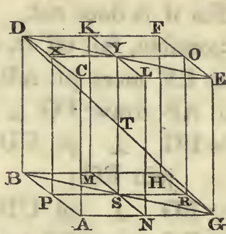
* This prop. Dr. Simson believes to be the addition of some editor.

PROP. XXXIX.—Theorem.

In a solid parallelopiped, if the sides of two of the opposite planes be divided, each into two equal parts, the common section of the planes passing through the points of division, and the diameter of the solid parallelopiped, cut each other into two equal parts.

Let the sides of the opp. pls. CF, AH of Sol. ▭ AF be ÷ into two equal parts in pts. K, L, M, N ; X, O, P, R ; and join KL, MN, XO, PR.



$\because$ DK = and ∥ CL,

$\therefore$ KL ∥ DC; 33. 1.

similarly { MN ∥ BA,
and BA ∥ DC.

Now $\because$ KL, BA ea. ∥ DC,

and not in the same pl. with it,

$\therefore$ KL ∥ BA: 9. 11.

and $\because$ KL, MN ea. ∥ BA,

and not in same pl. with it,

$\therefore$ KL ∥ MN; 9. 11.

$\therefore$ KL,

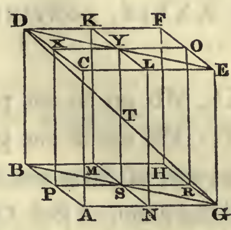
PROP. XXXIX. CONTINUED.

∴ KL, MN are in one pl.
similarly XO, PR are in one pl.
Let YS be the sec. of these pls. KN, XR;
and DG the diam. of Sol. □ AF.
Then shall YS and DG meet and cut ea. other into two = parts.

Join DY, YE, BS, SG;
∵ DX ∥ OE,
∴ alter. ∠ DXY = alter. ∠ YOE; 29. 1.
and ∵ DX = OE,
and XY = YO,
and contain equal ∠s,
∴ base DY = base YE,
and ∠ XYD = ∠ OYE; 4. 1.
∴ DYE is a rt. line; 14. 1.
similarly BSG is a rt. line;
and BS = SG.
And ∵ CA = and ∥ DB and EG,
∴ DB = and ∥ EG: 9. 11.
now DE, BG join their extrems.
∴ DE = and ∥ BG; 33. 1.
also DG, YS are drawn from pts. in one, to pts. in other,
and ∴ DG, YS are in same pl.
∴ it is manifest that DG, YS must meet;
let them meet in T;

and

PROP. XXXIX. CONTINUED.



and $\because$ DE $\parallel$ BG,

$\therefore$ alter. $\angle$ EDT = alter. $\angle$ BGT ; 29. 1.

and $\because$ also $\angle$ DTY = $\angle$ GTS, 15. 1.

$\therefore$ in $\triangle$s DTY, GTS ;

two $\angle$s in one = two $\angle$s in other,

and one side = one side,

viz. DY = GS,

(for they are the halves of DE, BG),

$\therefore$ DT = TG,

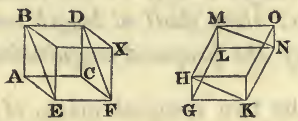
and YT = TS. 26.1.

Therefore if in a solid, &c. &c. Q. E. D.

PROP. XL.—THEOREM.

If there be two triangular prisms of the same altitude, the base of one of which is a parallelogram and the base of the other a triangle; if the parallelogram be double of the triangle, the prisms shall be equal to each other.

Let the prisms ABCDEF, GHKLMN be of same altitude, the first of which is contained by the two △s ABE, CDF, and the three □s AD, DE, EC; and the other by the two △s GHK, LMN, and the three □s LH, HN, NG; and let one of them have a □ AF, and the other a △ GHK for its base. And let □ AF = 2 △ GHK, the prism ABCDEF = prism GHKLMN.



Complete sols. AX, GO;

and ∵ □ AF	=	2 △ GHK,	
and □ HK	=	2 △ GHK,	34. 1.
∴ □ AF	=	□ HK;	
and conseq. sol. AX	=	sol. GO;	31.11.
now prism ABEDCF	=	½ sol. AX,	28.11.
and prism GHKLMN	=	½ sol. GO,	
∴ the prisms	=	ea. other.	

Wherefore, if there be two prisms, &c. &c. Q. E. D.

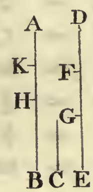
BOOK XII.

LEMMA I.

Which is the first proposition of the tenth book, and is necessary to some of the propositions of this book.

If from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half; and so on: there shall at length remain a magnitude less than the least of the proposed magnitudes.

Let AB and C be two unequal mags. of which AB > C. If from AB there be taken more than its half, and from the remainder more than its half, and so on; there shall at length remain a mag. < C.



For C may be multiplied so as to become > AB:
let DE be its mult. > AB;
and let DE be ÷ into DF, FG, GE. ea. = C;
from AB take BH > $\frac{1}{2}$ AB;
and from rem. AH take HK > $\frac{1}{2}$ AH,
&so on, until No. of divs. in AB = No. of divs. in DE;
and let the divs. in AB be AK, KH, HB;
and the divs. in DE be DF, FG, GE.

And

LEMMA I. CONTINUED.

And $\because$ DE	$>$	AB,
and that EG taken from DE	$\ngtr$	$\frac{1}{2}$ DE,
but that AH taken from AB	$>$	$\frac{1}{2}$ AB,
$\therefore$ rem. GD	$>$	rem. HA.
Again, $\because$ GD	$>$	HA,
and that GF	$\ngtr$	$\frac{1}{2}$ GD,
but HK	$>$	$\frac{1}{2}$ HA,
$\therefore$ rem. FD	$>$	AK :
and FD	$=$	C,
$\therefore$ AK	$<$	C.

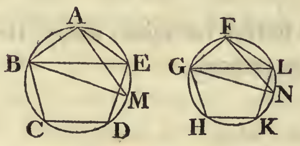
Q. E. D.

And if only the halves be taken away, the same thing may in the same way be proved.

PROP. I.

Similar polygons inscribed in circles, are to each other as the squares of their diameters.

Let ABCDE, FGHKL be two ⊙s, and in them the simil. polygons ABCDE, FGHKL; and let BM, GN be the diams. of the ⊙s. Then plgn. ABCDE : plgn. FGHLK :: BM^2 : GN^2.



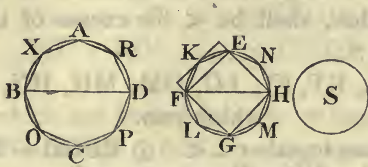
Join BE, AM, GL, FN.
And ∵ the plgns. simil. ea. other,
∴ △ABE is equiang. and simil. △FGL, 6. 6.
and ∴ ∠AEB = ∠FLG.
But ∠AEB = ∠AMB, 21. 3.
(for they are on same arc).
Similarly ∠FLG = ∠FNG;
∴ also ∠AMB = ∠FNG.
But rt. ∠ BAM = rt. ∠ GFN, 33. 1.
∴ rem. ∠s of △s ABM, FGN are = ea. other;
and ∴ △ABM is equiang. to △ FGN;
∴ BM : GN :: BA : GF; 4. 6.
and ∴ dupl. of BM : GN :: dupl. of BA : GF. 10 def. 5. & 22. 5.
But BM^2 : GN^2 :: dupl. of BM : GN; } 20. 6.
& plgn. ABCDE : FGHKL :: dupl. of BA : GF, }
∴ BM^2 : GN^2 :: plgn. ABCDE : plgn. FGHKL.

Wherefore, similar polygons, &c. &c. Q. E. D.

PROP. II.—Theorem.

Circles are to each other as the squares of their diameters.

Let ABCD, EFGH be two ⊙s, and BD, FH their diams.
Then as $BD^2 : FH^2$: : ⊙ ABCD : ⊙ EFGH.



For if it be not so,
then shall $BD^2 : FH^2$: : ⊙ ABD : some space < or > ⊙ EFGH.*

First—Let this space be S, < ⊙ EFGH;
and in ⊙ EFGH descr. sq. EG;
then sq. EG > $\frac{1}{2}$ of ⊙ EFGH;
for, if thro. pts. E, F, G, H there be drawn tangents to ⊙,
then shall sq. EG = $\frac{1}{2}$ sq. descr. about ⊙; 47. 1.
and the ⊙ < sq. descr. about it;
∴ sq. EG > $\frac{1}{2}$ of the ⊙.

Divide $\overset{\frown}{EF}$, $\overset{\frown}{FG}$, $\overset{\frown}{GH}$, $\overset{\frown}{HE}$ ea. into = parts in K, L, M, N;
join EK, KF, FL, LG, GM, MH, HN, NE;
∴ ea. of △s EKF, FLG, GMH, HNE > $\frac{1}{2}$ the seg. of ⊙, in which it stands;

* For there is some sq. = the ⊙ ABCD; let P be the side of it, and to three right lines BD, FH, and P, there can be a fourth proportional; let this be Q: therefore the sqs. of these four right lines are proportionals; that is, to the sqs. of BD, FH, and the ⊙ ABCD it is possible there may be a fourth proportional. Let this be S. And in like manner are to be understood some things in some of the following propositions.

PROP. II. CONTINUED.

for if tangents to ⊙ be drawn thro. K, L, M, N,
and □ s upon EF, FG, GH, HE be completed;

then ea. of △ s EKF, FLG, GMH, HNE } = ½ □ in which it is: 41. 1.

now every seg. is < □ in which it is,

∴ ea. of △ s EKF, FLG, GMH, HNE } > ½ seg. of ⊙ which contains it.

And if these arcs before named be ÷ ea. into two equal parts, and their extrems. be joined by rt. lines, by continuing to do this,* there will at length remain segments of the ⊙ which, together, shall be < the excess of the ⊙ EFGH above the space S. * Lemma.

Let the segs EK, KF, FL, LG, GM, MH, HN, NE be those which rem.

and are together < ⊙ EFGH — S;
∴ rest of ⊙, viz. plgn. EK....N > space S.

In the ⊙ ABCD,

describe plgn. AXB....R simil. plgn. EKF....N;

∴ $BD^2 : FH^2$ :: plgn. AX....R : plgn. EK....N; 1. 12.

but $BD^2 : FH^2$ :: ⊙ ABCD : S,

∴ ⊙ ABCD : S :: plgn. AX....R : plgn. EK....N: 11. 5.

but ⊙ ABCD > plgn. AX....R,

∴ space S > plgn. EK....N; 14. 5.

but it is also less, as was demon.

which is impossible.

∴ $BD^2 : FH^2$ is not as ⊙ ABCD : any space < ⊙ EFGH:
similarly $FH^2 : BD^2$ is not as ⊙ EFGH : any space < ⊙ ABCD.
Also $BD^2 : FH^2$ is not as ⊙ ABCD : any space > ⊙ EFGH;

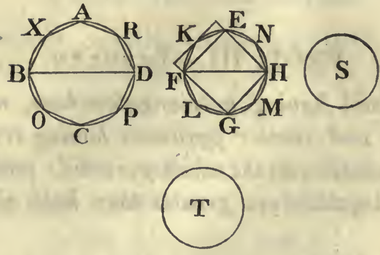
for if it be possible,

Secondly—Let it be to a space T, > ⊙ EFGH;

∴ invert. $FH^2 : BD^2$ :: T : ⊙ ABCD;

but

PROP. II. CONTINUED.



but $T : \odot ABCD :: \odot EFGH : $ a space $< \odot ABCD$,* 14. 5.

(for space $T > \odot EFGH$,) hyp.

$\therefore FH^2 : BD^2 :: \odot EFGH : $ a space $< \odot ABCD$;

which has been demon. to be imposs.

$\therefore BD^2 : FH^2$ is not as $\odot ABCD : $ any space $> \odot EFGH$;

and it has been demon.

that $BD^2 : FH^2$ is not as $\odot ABCD : $ any space $< \odot EFGH$.

$\therefore BD^2 : FH^2 :: \odot ABCD : \odot EFGH$.†

Wherefore, circles are, &c. &c. Q. E. D.

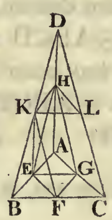
* For as, in the foregoing note, it was explained how it was possible, there could be a fourth proportional to the squares of BD, FH, and the circle ABCD, which was named S; so, in like manner, there can be a fourth proportional to this other space, named T, and the circles ABCD, EFGH. And the like is to be understood in some of the following propositions.

† Because, as a fourth proportional to the sqs. of BD, FH, and the $\odot$ AB CD, is possible, and that it can neither be $<$ nor $>$ $\odot$ EFGH, it must be $=$ to it.

PROP. III.—Theorem.

Every pyramid having a triangular base, may be divided into two equal and similar pyramids having triangular bases, and which are similar to the whole pyramid; and into two equal prisms which together are greater than half of the whole pyramid.

Let there be a pyramid whose base is the △ ABC and its vertex the pt. D. The pyr. ABCD can be ÷ into two equal and similar pyrs. having triangular bases, and similar to the whole; and into two equal prisms which together shall be > half of the whole pyr.



Divide AB, BC, CA, AD, DB, DC ea. into two equal parts in E, F, G, H, K, L;
and join EH, EG, GH, HK, KL, LH, EK, KF, FG:

∵ AE = EB,
and AH = HD,
∴ HE ∥ DB: 2. 6.
similarly HK ∥ AB,
∴ BH is a ▱;
and ∴ HK = EB: 34. 1.
but EB = AE,
∴ also AE = HK;
and AH = HD,
∴ EA, AH = KH, HD ea. to ea.
and ∠ EAH = KHD, 29. 1.
∴ base EH = base KD,
and △ AEH = & simil. △ HKD. 4. 1.

Similarly

PROP. III. CONTINUED.

Similarly $\triangle$ AGH = & simil. $\triangle$ HLD,

and $\because$ EH, HG which meet, are $\parallel$ KD, DL which meet,
but are not in same pl.

$\therefore \angle$ EHG = $\angle$ KDL. 10.11.

Again, $\because$ EH, HG = KD, DL, ea. to ea.

and that $\angle$ EHG = $\angle$ KDL,

$\therefore$ base EG = base KL, and $\triangle$ EHG = & simil. $\triangle$ KDL. 4. 1.

Similarly $\triangle$ AEG = & simil. $\triangle$ HKL;

$\therefore$ *pyr.* whose base is $\triangle$ AEG and vertex H = & simil. *pyr.* whose base is $\triangle$ KHL, and vertex D. C. 11.

And $\because$ HK $\parallel$ AB a side of $\triangle$ ADB,

$\therefore$ $\triangle$ ADB is equiang. to $\triangle$ HDK,
and their sides are propors. 4, 6.

$\therefore$ $\triangle$ ADB simil. $\triangle$ HDK.

Similarly $\triangle$ DBC simil. $\triangle$ DKL,

and $\triangle$ ADC simil. $\triangle$ HDL,

and also $\triangle$ ABC simil. $\triangle$ AEG.

But $\triangle$ AEG simil. $\triangle$ HKL, demon.

$\therefore$ $\triangle$ ABC simil. $\triangle$ HKL; 21. 6.

and *pyr.* whose base is $\triangle$ ABC, and vertex D, simil. *pyr.* whose base is $\triangle$ HKL, and vertex D; B. 11, and 11 def. 11.

but *pyr.* whose base is $\triangle$ AEG, and vertex H, simil. *pyr.* whose base is $\triangle$ HKL, and vertex D, demon.

$\therefore$ *pyr.* whose base is $\triangle$ ABC, and vertex D, simil. *pyr.* whose base is $\triangle$ AEG, and vertex H;

$\therefore$ ea. of *pyrs.* AEGH, HKLD simil. whl. *pyr.* ABCD.

And $\because$ BF = FC,

$\therefore \square$ BG = 2 $\triangle$ GFC; 41. 1.

and

PROP. III. CONTINUED.

and conseq. *prsm.* whose base is □ BG, and KH the rt. line opp. = *prsm.* whose base is △ GFC; and HKL the △ opp. 40.11.

(for they are of same alti. for pl. ABC ∥ pl. HKL.) 15.11.

and it is plain that ea. of the *prsms.* > either of *pyrs.* whose bases are △s AEG, HKL and vertices H, D;

for if EF be joined,

then *prsm.* whose base is □ BG; & KH the rt. line opp. > *pyr.* whose base is △ EBF, and vertex is K;

but this *pyr.* = *pyr.* whose base is △ AEG, and vertex H; C.11.

(for they are contained by equal and simil. pls.)

∴ *prsm*, whose base is □ BG, and KH the rt. line opp. > *pyr.* whose base is △ AEG, and vertex is H;

Now *prsm*, whose base is □ BG, and KH the rt. line opp. = *prsm*, whose base is △ GFC; & HKL the △ opp.

Also *pyr.* whose base is △ AEG, and vertex is H, = *pyr.* whose base is △ HKL, and vertex is D;

∴ the two *prsms.* > two *pyrs.* whose bases are △s AEG, HKL & vertices H, D.

∴ whl. *pyr.* ABCD is ÷ into two equal pyrs.; simil. to ea. other and the whl.

and also into two equal *prsms.*

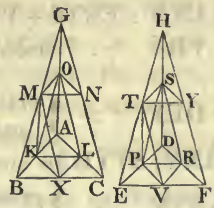
and the two prsms together > ½ whl. *pyr.*

Q. E. D.

PROP. IV.—Theorem.

If there be two pyramids of the same altitude, upon triangular bases, and each of them be divided into two equal pyramids similar to the whole pyramid, and also into two equal prisms; and if each of these pyramids be divided in the same manner as the first two, and so on: as the base of one of the first two pyramids is to the base of the other, so shall all the prisms in one of them be to all the prisms in the other, that are produced by the same number of divisions.

Let there be two pyramids of the same altitude, upon the triangular bases ABC, DEF, and having their vertices in pts. G, H; and let ea. be ÷ into two equal pyrs. similar to the whole, and into two equal prisms; and let ea. of the pyrs. thus made be conceived to be ÷ in the same manner, and so on. Then base ABC : base DEF, : : all *prisms* of pyr. ABCG : all *prisms* in pyr. DEFH made by same No. of divisions.



Make same constr. as in preceding.

And ∵ BX = XC,
and AL = LC,
∴ XL ∥ AB, 2. 6.
and △ ABC simil. △ LXC.
Similarly △ DEF simil. △ RVF.
And ∵ BC = 2 CX,
and EF = 2 FV,
∴ BC : CX : : EF : FV :

Now upon BC, CX are descr. the simil. rectilin. figs. ABC, LXC, and upon EF, FV are descr. simil. figs. DEF, RVF,

∴ △ ABC

PROP. IV. CONTINUED.

∴ △ ABC : △ LXC : : △ DEF : △ RVF; 22.6.
& permut. △ ABC : △ DEF : : △ LXC : △ RVF.
And ∵ pl. ABC ∥ pl. OMN,
and pl. DEF ∥ pl. STY, 15.11.
and that GC, HF are bisected in N, Y, by pls. OMN, STY,
∴ the ⊥s from G, H to bases ABC, DEF, (which, by hyp., are = ea. other,)
are cut into two equal parts by pls. OMN, STY,
and ∴ *prisms* LXCOMN, RVFSTY are same alti.

∴ base LXC : base RVF : : { prism LX....N : prism RV....Y;

i. e. △ ABC : △ DEF : : { prism LX....N : prism RV....Y. cor. 32.11.

And, ∵ two *prisms* of pyr. ABCD = ea. other,
and also two *prisms* of pyr. DEFH = ea. other,
∴ prism BLOM : prism LXN : : prism ERTS : prism VRY; 7.5.
∴ comp. BLOM + LXN : LXN : : ERTS + VRY : VRY;
and permut. BLOM + LXN : ERTS + VRY : : LXN : VRY:
but LXN : VRY : : base ABC : base DEF, demon.

∴ base ABC : base DEF : : { *prisms* in pyr. ABCG : *prisms* in pyr. DEFH.

And if pyrs. OMNG, STYH be similarly divided,
then base OMN : base STY : : { *prisms* in pyr. OMNG : *prisms* in pyr. STYH.
But base OMN : base STY : : base ABC : base DEF,
∴ base ABC : base DEF : : { *prisms* in pyr. ABCG : *prisms* in pyr. DEFH;
and so are *prisms* in pyr. OMNG : *prisms* in pyr. STYH,
and so are all *four* : all *four*.

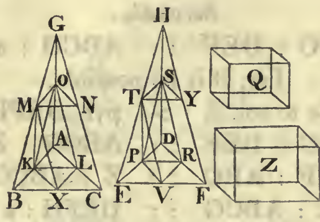
And the same may be demon. of prisms made by dividing the pyramids AKLO, DPRS, and also of all made by same No. of divisions.

Q. E. D.

PROP. V.—Theorem.

Pyramids of the same altitude which have triangular bases, are to each other as their bases.

Let the pyramids ABCG, DEFH be of same alti. Then base ABC : DEF :: ABCG : DEFH.



For, if it be not so, then

base ABC : base DEF :: ABCG : a sol. < or > DEFH.*

First—let it be to sol. Q < DEFH.

Divide pyr. DEFH into two equal pyrs. simil. to whole;

and also into two equal prisms,

then these two prisms > $\frac{1}{2}$ of the whl. pyr. 3. 12.

And, again, divide similarly the pyrs. made by this division, and so on,

until the *pyrs.* which rem. undiv. be *together* } < pyr. DEFH—sol. Q.

let these *pyrs.* be DPRS, STYH;

∴ the *prisms* which make the rest of pyr. DEFH } > sol. Q:

also div. ABCG, similarly, and into same No. of parts, as DEFH;

∴ base ABC : base DEF :: { *prisms* in ABCG : *prisms* in DEFH; 4. 12.

but ABC : DEF :: ABCG : Q,

∴ ABCG : Q :: { *prisms* in ABCG : *prisms* in DEFH;

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* This may be explained in the same way as at the note * in Prop. 2, in the like case.

but

PROP. V. CONTINUED.

but pyr. ABCG $>$ prisms contained in it,

$\therefore$ sol. Q $>$ prisms in DEFH;

but it is also less,

which is impossible:

$\therefore$ base ABC : base DEF is not as ABCG : any sol. $<$ DEFH:

Similarly DEF : ABC is not as DEFH : any sol. $<$ ABCG.

Secondly.

Neither is ABC : DEF :: ABCG : a sol. $>$ DEFH.

For, if it be possible,

let it be to sol. Z $>$ pyr. DEFH.

And $\because$ ABC : DEF :: ABCG : Z,

$\therefore$ invert. DEF : ABC :: Z : ABCG;

but Z : ABCG :: DEFH : a sol. $<$ ABCG,* 14. 5.

(for sol. Z $>$ pyr. DEFH),

$\therefore$ DEF : ABC :: DEFH : a sol. $<$ ABCG;

but the contrary to this has been proved,

$\therefore$ ABC : DEF is not as ABCG : a sol. $>$ DEFH,

and it has been proved,

that ABC : DEF is not as ABCG : a sol. $<$ DEFH,

$\therefore$ base ABC : base DEF :: pyr. ABCG : pyr. DEFH.

Wherefore pyramids, &c. &c. Q. E. D.

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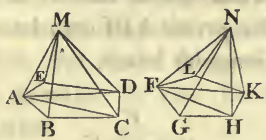
* This may be explained the same way as the like at the note * in Prop. 2.

PROP. VI.—Theorem.

Pyramids of the same altitude which have polygons for their bases, are to each other as their bases.

Let the pyrs. ABCDEM, FGHKLN be of the same altitude. Then base ABCDE : base FGHLK : : *pyr.* ABCDEM : pyr. FGHKLN.



Divide base ABCDE into △s ABC, ACD, ADE;
and base FGHKL into △s FGH, FHK, FKL;

and let the *No.* of pyrs. on bases ABC, ACD, ADE, whose com. ver. is M = the *No.* of pyrs. on bases FGH, FHK, FKL, whose com. ver. is N.

Then ∵ △ ABC : △ FGH : : pyr. ABCM : pyr. FGHN, 5. 12.

and △ ACD : △ FGH : : pyr. ACDM : pyr. FGHN,

and also △ ADE : △ FGH : : pyr. ADEM : pyr. FGHN,

∴ as all 1st *antecs.* : their com. *conseq.* : : all other *antecs.* : their com. *conseq.* 2 cor. 24. 5.

i. e. base ABCDE : base FGH : : pyr. ABCDEM : pyr. FGHN.

similarly base FGHKL : base FGH : : pyr. FGHKLN : pyr. FGHN.

∴ invert. base FGH : base FGHKL : : pyr. FGHN : pyr. FGHKLN.

Now ∵ base ABCDE : base FGH : : pyr. ABCDEM : pyr. FGHN,

and base FGH : base FGHKL : : pyr. FGHN : pyr. FGHKLN,

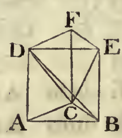
∴ ex æquali, base ABCDE : base FGHKL : : pyr. ABCDEM : pyr. FGHKLN. 22. 5.

Therefore pyramids, &c. &c. Q. E. D.

PROP. VII.—Theorem.

Every prism having a triangular base may be divided into three pyramids that have triangular bases, and are equal to each other.

Let there be a prism whose base is △ ABC and DEF the △ oppos. to it. The *prism* ABF can be ÷ into three equal *pyrs.* which have triangular bases.



Join BD, EC, CD;

Now, ∵ AE is a ▭,

and DB its diam.,

∴ △ ABD = △ EBD; 34. 1.

∴ *pyr.*, whose base is △ ABD, and vertex C, = *pyr.*, whose base is △ EBD, and vertex C; 5.12.

but *pyr.*, whose base is △ EBD, and vertex C, is same with *pyr.*, whose base is △ EBC, and vertex D;

(for they are contained by same pls.,)

∴ *pyr.*, whose base is △ ABD, and vertex C, = *pyr.*, whose base is △ EBC, and vertex D.

Again, ∵ FB is a ▭,

and CE its diam.,

∴ △ ECF = △ ECB; 34. 1.

∴ *pyr.*, whose base is △ ECB, and vertex D, = *pyr.*, whose base is △ ECF, and vertex D;

but *pyr.*, whose base is △ ECB, and vertex D, = *pyr.*, whose base is △ ABD, and vertex C; demon.

∴ *prism* ABF is ÷ into three equal *pyrs.* having △r bases; i. e. into *pyrs.* ABDC, EBDC, ECFD.

And

PROP. VII. CONTINUED.

And ∵ *pyr.*, whose base is △ ABD, and vertex C } is same with { *pyr.*, whose base is △ ABC, and vertex D,

(for they are contained by same pls.);

and that the *pyr.*, whose base is △ ABD, and vertex C, } = { $\frac{1}{3}$ of *prism* whose base is △ ABC, and DEF the opp. △, demon.

∴ *pyr.*, whose base is △ ABC, and vertex D, } = { $\frac{1}{3}$ of *prism* whose base is △ ABC, and DEF the opp. △.

Q. E. D.

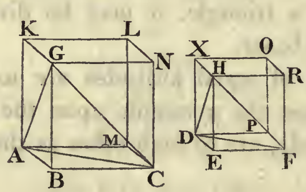
Cor. 1. From this it is manifest, that every pyramid is the third part of a prism which has the same base, and is of an equal altitude with it: for if the base of the prism be any other figure than a triangle, it may be divided into prisms having triangular bases.

Cor. 2. Prisms of equal altitudes are to one another as their bases; because the pyramids upon the same bases, and of the same altitude, are to each other as their bases.

PROP. VIII.—Theorem.

Similar pyramids, having triangular bases, are to each other in the triplicate ratio of that of their homologous sides.

Let the pyramids having △r bases ABC, DEF and their vertices the pts. G, H, be similar and similarly situated. The pyr. ABCG : pyr. DEFH : : tripl. of BC : EF.



Complete Sol. □ BL;
which is contd. by pls. BM, BN, BK and those opp.
Similarly compl. Sol. □ EO,
which is contd. by pls. EP, ER, EX and those opp.
and ∵ pyr. ABCG simil. pyr. DEFH,
∴ ∠ ABC = ∠ DEF,
∠ GBC = ∠ HEF, 11 def. 11.
and ∠ ABG = ∠ DEH:
and AB : BC : : DE : EF; 1 def. 6.
i. e. sides about the equal ∠ s are propors.;
∴ □ BM simil. □ EP.
Similarly { □ BN simil. □ ER,
and □ BK simil. □ EX;
∴ □ s BM, BN and BK simil. □ s EP, ER, EX;
but the 3 □ s BM, BN and BK = and simil. □ s opp. to them,
and the 3 □ s EP, ER and EX = and simil. □ s opp. to them, 24.11.

∴ *No.*

PROP. VIII. CONTINUED.

∴ *No.* of pls. which cont. sol. BL	=	*No.* of simil. pls. which cont. sol. EO;	
and their sol. ∠ s are	=	ea. other;	B. 11.
∴ sol. BL	simil.	sol. EO;	11def.11.
and ∴ sol. BK : sol. EO	: :	tripl. of BC : EF:	33.11.
but, ∵ *prsm.* in ea. sol.	=	$\frac{1}{2}$ sol.	28.11.
and that *pry.* in ea. prm.	=	$\frac{1}{3}$ prsm.	7. 12.
∴ prys. in ea. sol.	=	$\frac{1}{6}$ sol.	
and conseq. sol.BL : : sol.EO	: :	pyr.ABCG : pyr.DEFH,	15.5.
∴ pyr.ABCG : pyr. DEFH	: :	tripl. of BC : FE.	

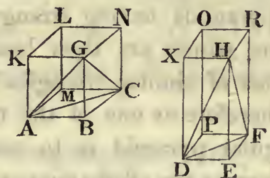
Q. E. D.

Cor. From this it is evident, that similar pyramids which have multangular bases, are likewise to each other in the triplicate ratio of their homologous sides. For they may be divided into similar pyramids having triangular bases, because the similar polygons, which are their bases, may be divided into the same number of similar triangles homologous to the whole polygons: therefore as one of the triangular pyramids in the first multangular pyramid is to one of the triangular pyramids in the other, so are all the triangular pyramids in the first to all the triangular pyramids in the other; that is, so is the first multangular pyramid to the other: but one triangular pyramid is to its similar triangular pyramid, in the triplicate ratio of their homologous sides; and therefore the first multangular pyramid has to the other, the triplicate ratio of that which one of the sides of the first has to the homologous side of the other.

PROP. IX.—Theorem.

The bases and altitudes of equal pyramids having triangular bases are reciprocally proportional: and triangular pyramids, of which the bases and altitudes are reciprocally proportional, are equal to each other.

First—Let the pyramids having $\triangle$r bases ABC, DEF and vertices G, H be = ea. other. Then the bases and altitudes of the pyramids shall be reciprocally proportional, viz. base ABC : base DEF : : *alti.* of pyr. DEFH : *alti.* of pyr. ABCG.



Complete sol. □ BL;
which is cont. by pls. AC, AG, GC and pls. opp.
also complete sol. □ EO;
which is cont. by pls. DF, DH, HF, and pls. opp.

And ∵ pyr. ABCG = pyr. DEFH,
and that sol. BL = 6 pyr. ABCG,
and sol. EO = 6 pyr. DEFH,
∴ sol. BL = sol. EO; 1 ax. 5.
and ∴ base BM : base EP : : *alti.* of EO : *alti.* of BL: 34. 11.
but base BM : base EP : : $\triangle$ ABC : $\triangle$ DEF, 15. 5.
∴ $\triangle$ ABC : $\triangle$ DEF : : *alti.* of EO : *alti.* of BL:
but *alti.* of sol. EO is same with *alti.* of pyr. DEFH,
also *alti.* of sol. BL is same with *alti.* of pyr. ABCG,
∴ base ABC : base DEF : : *alti.* of DEFH : *alti.* of ABCG.
∴ the bases and altis. of pyrs. ABCG, DEFH are reciprocally proportional.

Secondly—

PROP. IX. CONTINUED.

SECONDLY—Let the bases and alti. of pyramids ABCG, DEFH be reciprocally propor. viz. ABC : DEF : : *alti.* of DEFH : *alti.* of ABCG. Then shall pyr. ABCG = pyr. DEFH.

The same construction,

∵ base ABC : base DEF : : *alti.*of DEFH : *alti.*of ABCG,
and base ABC : base DEF : : □ BM : □ EP,
∴ □ BM : □ EP : : *alti.*of DEFH : *alti.*of ABCG;
but *alti.* of DEFH is same with *alti.* of sol. EO,
also *alti.* of ABCG is same with *alti.* of sol. BL,
∴ base BM : base EP : : *alti.* sol. EO : *alti.* of sol. BL;
i. e. bases and altis. of Sol. □ s are recip. propor.
∴ sol. BL = sol. EO. 34.11.
Now pyr. ABCG = $\frac{1}{6}$ BL,
and pyr. DEFH = $\frac{1}{6}$ EO,
∴ pyr ABCG = pyr. DEFH.

Wherefore, the bases, &c. &c. Q. E. D.

PROP. X.—Theorem.

Every cone is the third part of a cylinder which has the same base, and is of an equal altitude with it.

Let a cone have the same base with a cylinder, viz. the ⊙ ABCD, and the same alti. Then the cone = $\frac{1}{3}$ cyl. i.e. the cyl. = 3 cone.

If the cyl. ≠ 3 cone,
it is > or < 3 cone.
First, Let the cyl. > 3 cone;
descr. sq. AC in the ⊙;
then this sq. AC > $\frac{1}{2}$ of ⊙.*
On sq. AC, erect a *prsm*;
so that it be of same *alti.* with *cyl.*
then this *prsm.* > $\frac{1}{2}$ of *cyl.*
for, if a sq. be descr. about ⊙;
and a *prsm* erected on the sq. of same alti. as cyl.
then sq. AC = $\frac{1}{2}$ sq. circumscr.
and ∴ *prsm.* on sq. AC = $\frac{1}{2}$ of *prsm.* on circum. sq.
for they are to ea. other as their bases. 32.11.
Now cyl. < *prsm.* on circumscr. sq.
∴ *prsm.* on sq. AC of same alti. as cyl. > $\frac{1}{2}$ cyl.

Bis.

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* As was shown in Prop. II, of this Book.

PROP. X. CONTINUED.

Bis. $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, $\widehat{DA}$ in pts. E, F, G, H;
and join AE, EB, BF, FC, CG, GD, DH, HA.

then ea. of △s AEB, BFC, CGD, DHA } > ½ seg. in which it is. 2. 12.

Erect *prsms.* upon ea. of these △s of same alti. as cyl.
then shall ea. of these *prsms.* > ½ seg. of cyl. in which it is:
for, if thro. E, F, G, H, paralls. be drawn to AB, BC, CD, DA;
and □s be completed on the same AB, BC, CD, DA, and
sol. □s be erected on the □s.

then ea. of *prsms.* upon △s AEB, BFC, CGD, DHA } = ½ of its Sol. □ : 2 cor. 7. 12.

Now also *segs.* of cyl. on segs. of ⊙ cut off by AB, BC, CD, DA } < Sol. □s which cont. them,

∴ *prsms.* upon △s AEB, BFC, CGD, DHA } > { ½ segs. of cyl. in which they are :

∴, if ea. of the arcs be ÷ into two equal parts, and rt. lines be drawn from the pts. of division to the extrems. of the arcs, and upon the △s, thus made, prisms be erected of the same alti. with the cyl. and so on, there shall at length remain some *segs.* of the cyl. which together, shall be < cyl. — 3 cone. Lemma.

Let them be the *segs.* upon AE, EB, BF, FC, CG, GD, DH, HA,

∴ *rest of the cyl.* which is *the prsm.* whose base is the plgn. AEBFCGDH, & its alti. the same with that of cyl. } > 3 cone.

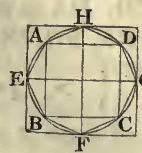
But this *prsm.* = { 3 *pyr.* on same base, whose ver. is same as the cone; 1 cor. 7. 12.

∴ *pyr.* on base AEBFCGDH, and of same vertex with cone } > cone, whose base is ⊙ ABCD;

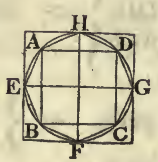
but this *pyr.* is cont. by the *cone*,
∴ also it is < cone;
which is impossible,
∴ the cyl. ≯ 3 cone.

 SECONDLY—

PROP. X. CONTINUED.



SECONDLY—Let the cyl. < 3 cone;
then the cone > $\frac{1}{3}$ cyl.
In ⊙ABCD descr. a sq. AC;
then sq. AC > $\frac{1}{2}$ ⊙;
And on the sq. AC erect a *pyr.* having same ver. as cone;
then this *pyr.* > $\frac{1}{2}$ cone;
for, as was before demon.
if a sq. be descr. about ⊙,
then sq. AC = $\frac{1}{2}$ this circumscr. sq.
and if, on these sqs. be erected *Sol.*□s of same alti. with cone, and which are also prsms.
then shall *prsm.* on sq. AC = $\frac{1}{2}$ *prsm.* upon circum. sq.
(for they are to ea. other as their bases); 32.11.
∴ *pyr.* whose base is sq. AC = $\frac{1}{2}$ *pyr.* whose base is the circumscr. sq.
But this last *pyr.* > cone which it contains,
∴ *pyr.* on sq. AC, whose vertex is that of the cone, > $\frac{1}{2}$ cone.



Bisect $\frown{AB}$, $\frown{BC}$, $\frown{CD}$, $\frown{DA}$ in pts. E, F, G, H:
and join AE, EB, BF, FC, CG, GD, DH, HA;
∴ ea. of △s AEB, BFC, CGD, DHA > $\frac{1}{2}$ seg. of ⊙ in which it is:
on ea. of these △s, erect *pyrs.* of same ver. with cone.
Then

PROP. X. CONTINUED.

Then *ea.* of these *pyrs.* > ½ seg. of cone in which it is; (as was before demon. of *prsms.* and segs. of *cyl.*)

And continuing these divisions, &c. there shall at length remain some *segs.* of the cone, which, together, shall be < cone — ⅓ cyl.

Let these be the segs. upon AE, EB, BF, FC, CG, GD, DH, HA;

∴ *rest of cone*, which is the *pyr.* whose base is plygn. AEBFCGDH and ver. same with cone, } > ⅓ cyl.

But this *pyr.* = { ⅓ *prsm.* on base AEBFCGDH, and of same alti. as cyl.

∴ this *prsm.* > *cyl.* whose base is ⊙ ABCD;

but this *prsm.* is cont. by the *cyl.*

which is absurd.

∴ The cyl. ≮ 3 cone;

and it has been proved;

that the cyl. ≯ 3 cone;

∴ cyl. = 3 cone;

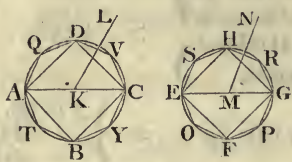
or cone = ⅓ cyl.

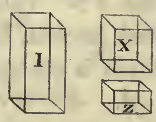
Wherefore, every cone, &c. &c. Q. E. D.

PROP. XI.—Theorem.

Cones and cylinders of the same altitude, are to each other as their bases.

Let the cones and cylinders having the ⊙s ABCD, EFGH their bases, and KL, MN their axes; and AC, EG, the diams. of their bases, be of the same altitude. Then ⊙ABCD : ⊙ EFGH : : cone AL : cone EN.





For, if it be not so
let ⊙ABCD : ⊙EFGH : : cone AL : a sol. $<$ or $>$ EN.
First—Let it be to a sol. X $<$ cone EN;
and let sol. Z $=$ cone EN $-$ sol. X,
∴ cone EN $=$ Z $+$ X:
in ⊙EFGH descr. a sq. FH,
then sq. FH $>$ $\frac{1}{2}$ ⊙,
on sq. FH erect a pyr. of same alti. with cone;
this *pyr.* shall be $>$ $\frac{1}{2}$ cone;
for, if a sq. be descr. about the ⊙,
and a *pyr.* be erected upon this sq. having same ver. as cone,*
then *pyr.* inscri. in cone $=$ $\frac{1}{2}$ *pyr.* circum. about cone;
(for they are to ea. other as their bases). 6.12.
But the cone $<$ circum. pyr.

∴ *pyr.*

---

* Vertex is put in place of altitude, which is in the Greek, because the pyramid, in what follows, is supposed to be circumscribed about the cone, and so must have the same vertex. And the same change is made in some places following.

PROP. XI. CONTINUED.

∴ *pyr.* whose base is sq. FH, and its vertex same as the cone. } > $\frac{1}{2}$ cone;

divide $\overset{\frown}{EF}$, $\overset{\frown}{FG}$, $\overset{\frown}{GH}$, $\overset{\frown}{HE}$ ea. into two equal parts in O, P, R, S;
and join EO, OF, FP, PG, GR, RH, HS, SE;

∴ ea. of △s EOF, FPG, GRH, HSE } > $\frac{1}{2}$ seg. in which it is:

on ea. of these △s erect a *pyr.* having same ver. with cone;
then ea. of these *pyrs.* > $\frac{1}{2}$ seg. of cone in which it is;
and by continuing these divisions, &c. there must at length remain some *segs.* of the cone which are together < sol. Z. Lemma.

Let these be the *segs.* on EO, OF, FP, PG, GR, RH, HS, SE,

∴ *rem. of cone*, viz. *pyr.* whose base is plgn. EOFPGRHS, and its ver. the same as the cone } > sol. X.

In ⊙ABCD,

descr. plgn. ATBYCVDQ simil. to plgn. EOFPGRHS;
and on AT...Q erect a *pyr.* with same ver. as cone AL.

and ∵ $AC^2 : EG^2$ :: AT...Q : EO...S. 1. 12.

and that $AC^2 : EG^2$ :: ⊙ABCD : ⊙EFGH, 2. 12.

∴ ⊙ABCD : ⊙EFGH :: plgn. AT...Q : plgn. EO...S; 11. 5.

but ⊙ABCD : ⊙EFGH :: cone AL : sol. X;

& plgn. AT...Q : plgn. EO...S } :: { *pyr.* whose base is AT...Q & vert. L, } : { *pyr.* whose base is EO...S, & vert. N, } 6. 12.

∴ cone AL : sol. X } :: { *pyr.* whose base is AT...Q & vertex L, } : { *pyr.* whose base is EO...S, and vertex N;

but cone AL > *pyr.* contained in it;

∴ sol. X > *pyr.* in cone EN; 14. 5.

but it was shewn that X < *pyr.* in cone EN,

which is absurd.

∴ ⊙ABCD is not to ⊙EFGH :: AL : any sol. < EN.

In same manner it may be demonstrated,

that ⊙EFGH is not to ⊙ABCD :: EN : a sol. < AL.

Neither

PROP. XI. CONTINUED.

Neither can

⊙ABCD : ⊙EFGH : : AL : a sol. $>$ EN.

For, if possible,

Secondly—Let it be so to sol. I $>$ cone EN;

∴ inver. ⊙EFGH : ⊙ABCD : : sol. I : cone AL;

but ∵ sol. I $>$ EN,

then sol. I : cone AL : : EN : a sol. $<$ AL; 14. 5.

∴ ⊙EFGH : ⊙ABCD : : EN : a sol. $<$ AL,

which was demon. to be impos.

∴ ⊙ABCD is not to ⊙EFGH : : AL : a sol. $>$ EN :

and it has been demon.

that ⊙ABCD is not to ⊙EFGH : : AL : a sol. $>$ EN :

∴ ⊙ABCD : ⊙EFGH : : cone AL : cone EN :

but cone : cone : : cylinder : cylinder, 15. 5.

for the cyls. = 3 cone ea. to ea. 10.12.

∴ ⊙ABCD : ⊙EFGH so are cyls. upon them of same alti.

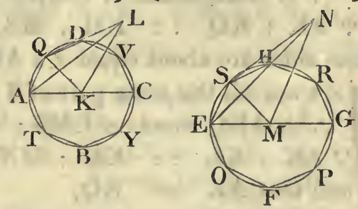
Wherefore cones and cylinders, &c. &c. Q. E. D.

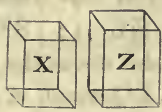
PROP. XII.—Theorem.

Similar cones and cylinders have to each other the triplicate ratio of that which the diameters of their bases have.

Let the cones and cylinders having ⊙s ABCD, EFGH for their bases, and the diams. of their bases AC, EG; and KL, MN axes of cones or cyls. be similar to ea. other.

$$\text{Then}\left\{\begin{array}{l}\textit{Cone}\text{ whose base}\\ \text{is ABCD, and}\\ \text{vert. pt. L}\end{array}\right\} : \left\{\begin{array}{l}\textit{Cone}\text{ whose base}\\ \text{is EFGH, and}\\ \text{vert. N}\end{array}\right\} :: \left\{\begin{array}{l}\text{tripl. of}\\ \text{AC : EG.}\end{array}\right.$$





For if not,

$$\text{Then cone ABCDL} : \left\{\begin{array}{l}\text{some solid}\\ <\text{or}>\text{cone}\\ \text{EFGHN}\end{array}\right\} :: \text{tripl. of AC : EG.}$$

First—Let it have it to sol. X < cone EFGHN;

make same constr. as in the preceding proposition;

and it may be demon., similarly as in that prop.;

$$\text{that}\left\{\begin{array}{l}\textit{pyr.}\text{ whose base is}\\ \text{plygn. EOFPGR}\\ \text{HS and vert. N}\end{array}\right\} > \text{sol. X.}$$

In ⊙ ABCD

descr. plygn. ATBYCVDQ simil. plygn. EOFPGRHS;

on ATB....Q erect a *pyr.* with same ver. as cone;

and let LAQ be one of △s contg. pyr. on ATB....Q, whose ver. is L;

and let NES be one of △s contg. pyr. on EOF....S, whose ver. is N;

join KQ, MS:

then,

PROP. XII. CONTINUED.

then, $\because$ cone ABCDL simil. cone EFGHN,

$\therefore$ AC : EG :: axis KL : ax. MN; 24 def. 11.

and AC : EG :: AK : EM, 15. 5.

$\therefore$ AK : EM :: ax. KL : ax. MN;

and alternato. AK : KL :: EM : MN;

and rt. $\angle$ AKL = rt. $\angle$ EMN:

and $\because$ the sides about these equal $\angle$ s are propors.,

$\therefore$ $\triangle$ AKL simil. $\triangle$ EMN. 6. 6.

Again, $\because$ AK : KQ :: EM : MS,

and that these sides are about equal $\angle$ s AKQ, EMS,

(for these $\angle$ s are ea. the same part of 4 rt. $\angle$ s),

$\therefore$ $\triangle$ AKQ simil. $\triangle$ EMS: 6. 6.

and $\because$ AK : KL :: EM : MN, demon.

and that AK = KQ,

and EM = MS,

$\therefore$ QK : KL :: SM : MN:

and $\because$ these are the sides about the rt. $\angle$ s QKL, SMN,

$\therefore$ $\triangle$ LKQ simil. $\triangle$ NMS:

and $\because$ $\triangle$ AKL simil. $\triangle$ EMN,

$\therefore$ LA : AK :: NE : EM;

and $\because$ $\triangle$ AKQ simil. EMS,

$\therefore$ KA : AQ :: ME : ES;

$\therefore$ ex æquali LA : AQ :: NE : ES. 22. 5.

Again, $\because$ $\triangle$ LQK simil. $\triangle$ NSM,

$\therefore$ LQ : QK :: NS : SM;

and $\because$ $\triangle$ KAQ simil. $\triangle$ MES,

$\therefore$ KQ : QA :: MS : SE;

$\therefore$ ex æquali LQ : QA :: NS : SE; 22. 5.

& it was proved that QA : AL :: SE : EN;

$\therefore$ again ex æquali QL : LA :: SN : NE;

and these are the sides about $\triangle$ s LQA, NSE,

$\therefore$ $\triangle$ LQA is equiang. and simil. $\triangle$ NSE; 5. 6.

and $\therefore$ *pyr.* whose base is $\triangle$ AKQ and ver. L } simil. { *pyr.* whose base is $\triangle$ EMS and ver. N,

(for their sol. $\angle$ s = ea. other; and are contd. by same No. of pls.). B. 11.

Now

PROP. XII. CONTINUED.

Now ∵ pyr. AKQL simil. pyr. EMSN,
and that they have △r bases,
∴ pyr. AKQL : pyr. EMSN : : tripl. of AK : homol. side EM; 8. 12.

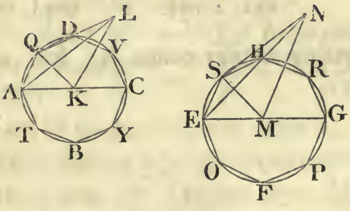
similarly, if rt. lines be drawn from D, V, C, Y, B, T to K;
and from H, R, G, P, F, O to M;
and if *pyrs.* be erected on the △s with vertices of the cones;
it may be demon., that
ea. pyr. in first cone has to ea. in the other, taking them in same order, the triplicate of AK : EM, i. e. the tripl. of AC : EG;
But one antec. : its conseq. : : all the antecs. : all conseqs.; 12. 5.

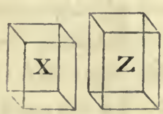
$$\therefore \left.\begin{array}{l} pyr.\ \text{AKQL} : \\ \text{pyr. EMSN} \end{array}\right\} : : \left\{\begin{array}{l} \text{whl. } pyr. \text{ whose} \\ \text{base is plygn.} \\ \text{DQA....V} \\ \text{and ver. L,} \end{array}\right\} : \left\{\begin{array}{l} \text{whl. } pyr. \text{ whose} \\ \text{base is plygn.} \\ \text{HSE....R} \\ \text{and ver. N;} \end{array}\right.$$

$$\therefore \text{also} \left\{\begin{array}{l} pyr. \text{ whose base} \\ \text{is DQA....V} \\ \text{and ver. L} \end{array}\right\} : \left\{\begin{array}{l} \text{pyr. whose base} \\ \text{is HSE....R} \\ \text{and ver. N} \end{array}\right\} : : \left\{\begin{array}{l} \text{tripl. of} \\ \text{AC : EG;} \end{array}\right.$$

$$\text{but} \left\{\begin{array}{l} cone, \text{ whose base} \\ \text{is } \odot \text{ ABCD} \\ \text{and ver. L} \end{array}\right\} : \text{sol. X} : : \text{tripl. of AC : EG;}$$ hyp.

$$\therefore \left.\begin{array}{l} \text{cone ABCDL} \\ : \text{sol. X} \end{array}\right\} : : \left\{\begin{array}{l} pyr. \text{ whose base} \\ \text{is DQA....V} \\ \text{and ver. L} \end{array}\right\} : \left\{\begin{array}{l} pyr. \text{ whose base} \\ \text{is HSE....R} \\ \text{and ver. N:} \end{array}\right.$$





But cone ABCDL > *pyr.* contained in it,

$$\therefore \text{sol. X} > \left\{\begin{array}{l} pyr. \text{ whose base is} \\ \text{HSE....R and} \\ \text{ver. N;} \end{array}\right\}$$ 14.5.

but it is also less,
which is impossible.

PROP. XII. CONTINUED.

∴ Cone ABCDL has not to a sol. < cone EFGHN the tripl. of AC : EG.

Similarly it may be demon.,

that neither is cone EFGHN : a sol. < cone ABCDL : : tripl. of EG : AC.

Nor is cone ABCDL : a sol. > cone EFGHN : : tripl. of AC : : EG

for if it be possible,

Secondly—Let it have to it a sol. Z > cone EFGHN ;

∴ inver. sol. Z : cone ABCDL : : tripl. of EG : AC ;

but sol. Z : cone ABCDL : : { cone EFGHN : *a sol.* < cone ABCDL, 14.5.

(for sol. Z > cone EFGHN),

∴ EFGHN : a sol. < ABCDL : : tripl. of EG : AC ;

which was demon. to be impossible :

∴ ABCDL has not to a sol. > EFGHN the tripl. of AC : EG.

And it was demonstrated,

that ABCDL has not to a sol. < EFGHN the tripl. of AC : EG.

∴ cone ABCDL : cone EFGHN : : tripl. of AC : EG ;

but cone : cone : : cyl. : cyl., 15.5.

(for every cone = { $\frac{1}{3}$ cyl. on same base and alti.),

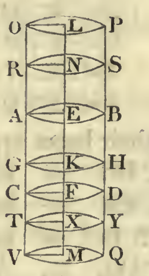
∴ cyl. : cyl. : : tripl. of AC : EG.

Wherefore similar cones, &c. &c. Q. E. D.

PROP. XIII.—Theorem.

If a cylinder be cut by a plane parallel to its opposite planes, or bases, it divides the cylinder into two cylinders, one of which is to the other as the axis of the first to the axis of the other.

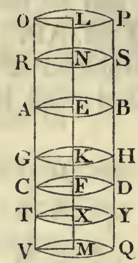
Let the cyl. AD be cut by the pl. GH ∥ to opp. pls. AB, CD, meeting ax. EF in pl. K, and let the line GH be the sec. of pl. GH and the surface of cyl. AD. Let CE be a □, in any position of it, by the revolution of which about the rt. line EF, the cyl. AD is described; and let GK be the sec. of pl. GH, and the pl. CE.



∵ parall. pls. AB, GH are cut by pl. AK,
∴ their com. sec. ∥ ea. other;
i. e. AE ∥ KG; 16.11.
∴ AK is a □,
and GK = EA from cent. of ⊙ AB:
similarly ea. of rt. lines from K to GH } = { rt. lines from cent. of ⊙ AB to ○,
and ∴ all of them = ea. other;
∴ line GH is the arc of a ⊙ whose centre is K, 15 def. 5.
∴ pl. GH divides cyl. AD into cyls. AH, GD;
for they are the same which would be described by the revolution of the □s AK, GF about the rt. lines EK, KF.
It is to be shewn that cyl. AH : cyl. HC : : ax. EK : ax. EF.
Produce the axis EF both ways;
and take any No. of rt. lines EN, NL, ea. = EK;
and any No. of rt. lines FX, XM ea. = FK;
and let pls. ∥ to AB, CD pass thro. pts. L, N, X, M:
∴ secs.

PROP. XIII. CONTINUED.

∴ secs. of these pls. with surface of cyl. produced are ⊙s
whose cents. are L, N, X, M;
as was proved of the pl. GH;
and these pls. shall cut off cyls. PR, RB, DT, TQ.
And ∵ axs. LN, NE, EK = ea. other,
∴ cyls. PR, RB, BG are to ea. other as their bases. 11.12.



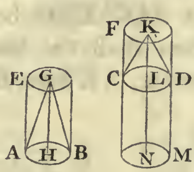
But their bases are equal,
∴ cyls. PR, RB, BG = ea. other.
and ∵ axs. LN, NE, EK = ea. other,
and that also cyls. PR, RB, BG = ea. other,
and that *No.* of axs. = *No.* of cyls.,
∴ cyl. PG is same mult. of cyl. GB that ax. KL is of ax. KE;
similarly cyl. QG is same mult. of cyl. GD that ax. MK is of ax. KF;
and if ax. KL = ax. KM,
then cyl. PG = cyl. GQ;
and if greater, greater; if less, less.
Now, ∵ there are four mags. EK, KF, BG, GD,
and that ax. KL, and cyl. PG are any equimults. of ax. EK and cyl. BG,
and that ax. KM, and cyl. GQ are any equimults. of ax. KF and cyl. GD,
and that if KL > KM,
then PG > GQ,
if equal, equal; if less, less.
∴ ax. EK : ax. KF : : cyl. BG : cyl. GD. 5 def. 5.

Wherefore if a cylinder, &c. &c. Q. E. D.

PROP. XIV.—Theorem.

Cones and cylinders upon equal bases are to each other as their altitudes.

Let the cyls. EB, FD be upon equal bases AB, CD. Then shall cyl. EB : cyl. FD : : ax. GH : ax. KL.



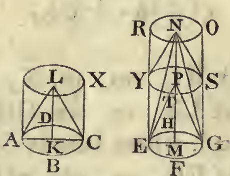
Prod. ax KL to pt. N;
and make LN = ax. GH;
and let CM be a cyl. whose base is CD and ax. LN;
and ∵ *alti.* of EB = *alti.* of CM,
these cyls. are to ea. other as their bases; 11. 12.
but their bases are equal,
∴ cyl. EB = cyl. CM,
And ∵ cyl. FM is cut by pl. CD ‖ to opp. pls.
∴ cyl. CM : cyl. FD : : ax. LN : ax. KL; 13. 12.
but cyl. CM = cyl. EB,
and ax. LN = ax. GH,
∴ cyl. EB : cyl. FD : : ax. GH : ax. KL:
and ∵ the cyls. = 3 cone,
∴ cyl. EB : cyl. FD : : cone ABG : cone CDK. 15. 5.
and ∴ ax. GH : ax. KL : : cone ABG : cone CDK : : cyl. EB : cyl. FD.

Wherefore cones, &c. &c. Q. E. D.

PROP. XV.—Theorem.

The bases and altitudes of equal cones and cylinders are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

First—Let ⊙s BD, FH, whose diams. are AC, EG, be the bases, and KL, MN the axes, as also the altis. of equal cones and cylinders; and let ALC, ENG be the cones, and AX, EO the cylinders. Then shall the bases and altis. of cyls. AX, EO be recip. propor. i. e. base BD : base FH : : alti. MN : alti. KL.



Either the alti. MN is = or ≠ alti. KL.
First—Let MN = KL;
and ∵ also cyl. AX = cyl. EO,
and that cones and cyls. of = alti. are to ea. other as their bases, 11.12.
∴ base ABCD = base EFGH; A. 5.
and base BD : base FH : : alti. MN : alti. KL.
Secondly—Let alti. MN ≠ alti. KL;
and let MN > KL,
from MN take MP = KL;
and thro. P, cut cyl. EO by pl. TYS ∥ to opp. pls. of ⊙s HF, RO;
∴ sec. of pl. TYS and surface of cyl. EO shall be a ⊙;
and ES is a cyl. whose base is ⊙ HF and alti. MP.
And ∵ cyl. AX = cyl. EO,
∴ AX : cyl. ES : : cyl. EO : cyl. ES; 7.5.
but AX : ES : : base BD : base FH, 11.12.
(for *alti.* of AX = *alti.* of ES),

and

PROP. XV. CONTINUED.

and cyl. EO : cyl. ES : : alti. MN : alti. MP, 13.12.
(for *cyl.* EO is cut by pl. TYS ∥ its opp. pls.),
∴ base BD : base FH : : alti. MN : alti. MP;
but MP = KL,
∴ base BD : base FH : : alti. MN : alti. KL;
i. e. the bases and altis. of equal cyls. are recip. propor.

SECONDLY—Let the bases and altitudes of the cylinders AX, EO be recip. propor., viz. base BD : base FH : : alti. MN : alti. KL. Then the cyl. AX = cyl. EO.

First—Let base BD = base FH,
then ∵ base BD : base FH : : alti. MN : alti. KL,
∴ MN = KL A. 5.
and ∴ cyl. AX = cyl. EO. 11.11.

Secondly—Let base BD ≠ base FH,
and let BD > FH
and ∵ BD : FH : : MN : KL,
∴ MN > KL. A. 5.

The same constr. being made;
∵ base BD : base FH : : alti. MN : alti. KL,
and ∵ alti. KL = alti. MP,
∴ base BD : base FH : : cyl. AX : cyl. ES; 11.12.
and alti. MN : alti. MP or KL : : cyl. EO : cyl. ES;
∴ cyl. AX : cyl. ES : : cyl. EO : cyl. ES.
∴ cyl. AX = cyl. EO.

And the same reasoning holds in cones.

Q. E. D.

PROP. XVI.—Problem.

In the greater of two circles that have the same centre, to inscribe a polygon of an even number of equal sides, that shall not meet the lesser circle.

Let ABCD, EFGH be two given ⊙s having same cent. K. It is required to inscribe in the greater ⊙ ABCD a polygon of an even number of equal sides, that shall not meet the lesser ⊙.

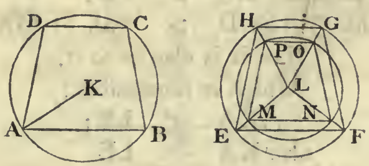
Thro. K draw rt. line BD;
and from G, where it meets ○ of lesser ⊙,
draw GA rt. ∠s to BD;
and prod. GA to C;
∴ AC touches ⊙EFGH. 16. 3.
Then, if $\widehat{BAD}$ be bisec. continually, there shall at length remain an *arc* < $\widehat{AD}$. Lemma.
Let this be $\widehat{LD}$;
and from L draw LM ⊥ BD;
and prod. LM to N;
Join LD, DN:
∴ LD = DN:
and ∵ LN ∥ AC,
and that AC touches ⊙ EFGH,
∴ LN shall not meet ⊙ EFGH;
and much less shall rt. lines LD, DN meet it.

So that, if rt. lines = LD be appl. in ⊙ ABCD there shall be described in the ⊙ a polygon of an even No. of equal sides that shall not meet the lesser ⊙.

Q. E. F.

LEMMA II.

If two trapeziums ABCD, EFGH be inscribed in the circles, the centres of which are the points K, L; and if the sides AB, CD be parallel, as also EF, HG; and the other four sides AD, BC, EH, FG, be all equal to each other; but the side AB greater than EF, and DC greater than HG; the right line KA from the centre of the circle in which the greater sides are, is greater than the right line LE drawn from the centre to the circumference of the other circle.



If it be possible,
let KA $\ngtr$ LE;
then KA must be either $=$ or $<$ LE.
First—Let KA $=$ LE;
then in the two equal ⊙s,
$\because$ AD, BC in one $=$ EH, FG in other,
$\therefore$ $\widehat{AD}$, $\widehat{BC}$ $=$ $\widehat{EH}$, $\widehat{FG}$; 28. 3.
but $\because$ AB, DC $>$ EF, GH ea. than ea.
$\therefore$ $\widehat{AB}$, $\widehat{DC}$ $>$ $\widehat{EF}$, $\widehat{GH}$;
$\therefore$ whl. ○ABCD $>$ whl. ○EFGH:
but it is also $=$ to it,
which is impossible:
$\therefore$ KA $\neq$ LE.
Secondly—Let KA $<$ LE;
and make LM $=$ KA;
and with cent. L and dist. LM, descr. ⊙ MNOP,
meeting rt. lines LE, LF, LG, LH, in M, N, O, P;

 and

LEMMA II. CONTINUED.

and join MN, NO, OP, PM,

which are respectively $\parallel$ & $<$ EF, FG, GH, HE. 2. 6.

Now, $\because$ EH $>$ MP,

$\therefore$ AD $>$ MP;

and $\odot$ ABCD $=$ $\odot$ MNOP,

$\therefore \overset{\frown}{AD} > \overset{\frown}{MP}$;

similarly $\overset{\frown}{BC} > \overset{\frown}{NO}$;

and $\because$ AB $>$ EF,

and that EF $>$ MN,

much more $\therefore$ AB $>$ MN;

$\therefore \overset{\frown}{AB} > \overset{\frown}{MN}$:

similarly $\overset{\frown}{DC} > \overset{\frown}{PO}$,

$\therefore$ whl. $\bigcirc$ ABCD $>$ whl. $\bigcirc$ MNOP;

but it is also $=$ to it,

which is impossible;

$\therefore$ KA $\not<$ LE;

also KA $\neq$ LE,

$\therefore$ KA $>$ LE.

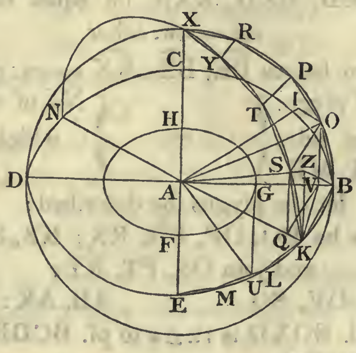
Q. E. D.

Cor. And if there be an isosceles $\triangle$ whose sides are $=$ AD, BC, but its base $<$ AB which is $>$ DC; then KA shall, in same manner, be demon. to be $>$ than the rt. line from the cent. to $\bigcirc$ of the $\odot$ described about the $\triangle$.

PROP. XVII.—Problem.

In the greater of two spheres which have the same centre, to inscribe a solid polyhedron, the superficies of which shall not meet the lesser sphere.

Let there be two spheres about same cent. A; it is required to describe in the greater a solid polyhedron whose superficies shall not meet the lesser sphere.



Let the spheres be cut by a pl. passing thro. the cent., then the com. secs. of it with the spheres shall be ⊙s; because the sphere is described by the revolution of a ½ ⊙ about the diam. remaining immoveable; so that in whatever position the ½ ⊙ be conceived, the com. sec. of the pl. in which it is with the superficies of the sphere is the ○ of a ⊙; and this is a great ⊙ of the sphere, because the diam. of the sphere, which is also the diam. of the ⊙, is >* any rt. line in the ⊙ or sphere. * 15. 3.

Then let ⊙ made by sec. of pl. with greater sph. be BCDE,
and that made by sec. of pl. with lesser sph. be FGH,
and draw diam. BD rt. ∠s to diam. CE;

In ⊙ BCDE, descr. a plygn. of an even No. of equal sides not meeting lesser ⊙ FGH; 16.12.

let its sides in $\frown{BE}$, which = ¼ ⊙, be BK, KL, LM, ME;
join KA, and prod. it to N;
from A draw AX rt. ∠s to pl. of ⊙ BCDE,
so that AX meet superf. of sph. in X;

and

PROP. XVII. CONTINUED.

and let pls. pass thro. AX and ea. of rt. lines BD, KN,
which pls. shall prod. great ⊙s in superf. of sphs.;
and let BXD, KXN be the ½ ⊙s thus made on dias. BD, KN:
then ∵ XA is rt.∠s to pl. of ⊙ BCDE,
∴ every pl. thro. XA is rt.∠s to pl. of ⊙ BCDE; 18. 11.
and ∴ ½ ⊙s BXD, KXN are rt.∠s to pl. of ⊙ BCDE.
And ∵ ½ ⊙s BED, BXD, KXN, on equal dias. BD, KN, are = ea. other,
∴ their halves $\widehat{BE}$, $\widehat{BX}$, $\widehat{KX}$ = ea. other.

∴ No. of sides of plygn. in $\widehat{BX}$, $\widehat{KX}$ = sides BK, KL, LM, ME } = { No. of sides of plygn. which are in $\widehat{BE}$;

let the plygns. be described;
and their sides be BO, OP, PR, RX; KS, ST, TY, YX;
and join OS, PT, RY;
from O, S draw OV, SQ ⊥ AB, AK:
and ∵ pl. BOXD is rt.∠s to pl. BCDE,
and that in one BOXD, is drawn OV } ⊥ AB com. sec. of pls.,
∴ OV ⊥ pl. BCDE: 4 def. 11.
similarly SQ ⊥ pl. BCDE,
for pl. KSXN is rt.∠s to pl. BCDE.

Join VQ;
and ∵ in the equal ½ ⊙s BXD, KXN,
that $\widehat{BO}$ = $\widehat{KS}$,
and OV, SQ ⊥ their diams.,
∴ OV = SQ, 26. 1.
and BV = KQ:
But whl. BA = whl. KA,
∴ rem. VA = rem. QA;
∴ BV : VA : : KQ : QA;
∴ VQ ∥ BK: 2. 6.
and ∵ ea. of OV, SQ is rt.∠s to pl. of ⊙ BCDE,
∴ OV ∥ SQ; 6. 11.
and also OV = SQ, demon.

and

PROP. XVII. CONTINUED.

∴ QV = and ∥ SO; 33. 1.

and ∵ QV ∥ SO and KB,

∴ OS ∥ KB; 9. 11.

and ∴ BO, KS which join them are in same pl. with the ∥ s,

and quadrilat. fig. KBOS is in one pl.;

and, if PB, TK be joined,

and from P, F be drawn, rt. lines ⊥ to AB, AK;

it may be demon.

that TP ∥ KB;

similarly as was demon. SO ∥ KB,

∴ TP ∥ SO; 9. 11.

and quadrilat. fig. SOPT is in one pl.

similarly quadrilat. fig. TPRY is in one pl.

and fig. YRX is in one pl. 2. 11.

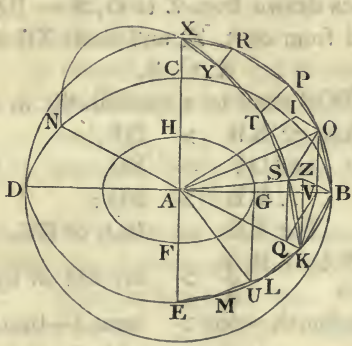
∴ If from O, S, P, T, R, Y be drawn rt. lines to A,

there shall be formed *a sol. polyhed.* between $\widehat{BX}$, $\widehat{KX}$,

and composed of pyrs. whose bases are KBOS, SOPT,

TPRY, YRX, ∴

and of which pyrs., A is the com. ver.



And if the same construction be made upon ea. of the sides KL, LM, ME, which has been made upon BK, and the same also be done in the other three quadrants, and in the other hemisphere; *there shall be formed a solid polyhedron described in the sphere,* composed of pyrs., the bases of which are

PROP. XVII. CONTINUED.

are the aforesaid quadrilat. figs., and △ YRX, and those formed in same manner in the rest of the sphere, the com. ver. of them all being the pt. A.

And *the superficies of this solid polyhedron does not meet the lesser sphere* in which is the ⊙ FGH. For,

From A draw AZ ⊥ pl.fig. KBOS meeting it in Z; 11. 11.

and join BZ, ZK.

And ∵ AZ ⊥ pl. KBOS,

∴ AZ ⊥ BZ, and ZK : 3 def. 11.

and ∴ AB = AK,

and that $AZ^2+ZB^2 = AB^2$,
and $AZ^2+ZK^2 = AK^2$, } 47. 1.

∴ $AZ^2+ZB^2 = AZ^2+ZK^2$.

Take away com. AZ^2;

∴ $BZ^2 = ZK^2$;

and ∴ BZ = ZK :

similarly it may be demon.,

that rt. lines drawn from Z to O, S = BZ or ZK,

∴ a ⊙ described from cent. Z, and dist. ZB shall pass thro. K, O, S,

and KBOS shall be a quadril. fig. in a ⊙.

And ∵ KB > QV,

and QV = SO,

∴ KB > SO :

but KB = BO, or KS,

∴ *ea. arc*, cut off by KB, BO, KS is } > *arc* cut off by OS ;

and *these 3 arcs* + a fourth = one > *same* 3 + that cut off by OS ;

i.e. > whl. ○ of ⊙ ;

∴ *arc* subtended by KB > ¼ whl. ○ of ⊙ KBOS ;

and conseq. ∠ BZK at cent. > rt. ∠.

And ∵ ∠ BZK > rt. ∠,

∴ $BK^2 > BZ^2+ZK^2$; 12. 2.

i.e. $BK^2 > 2\,BZ^2$.

Join

PROP. XVII. CONTINUED.

Join KV,
and in △s KBV, OBV,
∴ KB, BV = OB, BV, ea. to ea.
and that they cont. equal ∠ s,
∴ ∠ KVB = ∠ OVB: 4. 1.
but ∠ OVB is a rt. ∠,
∴ also ∠ KVB is a rt. ∠.
And ∵ BD < 2 DV,
∴ DB × BV < 2 DV × VB;
i. e. KB^2 < 2 KV^2,
but KB^2 > 2 BZ^2,
∴ KV^2 > BZ^2.
And ∵ BA = AK,
and that $BZ^2 + ZA^2$ = BA^2,
and $KV^2 + VA^2$ = AK^2,
∴ $BZ^2 + ZA^2$ = $KV^2 + VA^2$;
and of these,
KV^2 > BZ^2,
∴ VA^2 < ZA^2;
and AZ > VA;
much more than AZ > AG:
∵ in preced. prop. it was shewn,
that KV falls without ⊙ FGH;
and AZ ⊥ pl. KBOS;
and is ∴ < all rt. lines which can be drawn from A the cent. of sph. to that pl.

∴ *The pl. KBOS does not meet the lesser sphere.*

And also *the other pls. between quadrants BX, KX, do not meet the lesser sph.* for

From A, draw AI ⊥ pl. of quadril. fig SOPT,
join IO;
and as was demon. of pl. KBOS and pt. Z, similarly it may be shewn,
that pt. I is the cent. of ⊙ descr. about SOPT:
and that OS > PT;
and it was shewn that PT ∥ OS.

Now

PROP. XVII. CONTINUED.

Now ∵ the two trapezs. KBOS, SOPT inscr. in ⊙s have parallel sides,

viz. { BK ∥ OS, and OS ∥ PT,

and that their other sides, BO, KS, OP, ST } = ea. other,

and that BK > OS,

and OS > PT,

∴ ZB > IO. 2 Lemma 12.

Join AO,

which will = AB;

and ∴ AIO, AZB are rt. ∠ s,

∴ $AI^2 + IO^2 = AO^2$ or AB^2;

i. e. $AI^2 + IO^2 = AZ^2 + ZB^2$;

and $ZB^2 > IO^2$.

∴ $AZ^2 < AI^2$

and AZ < AI;

and it was proved AZ > AG;

much more than AI > AG.

∴ *Pl. SOPT falls wholly without the lesser sphere.*

In same way it may be demon.

that *pl. TPRY falls wholly without lesser sphere;*

and *also pl. △ YRX falls wholly without lesser sphere;* cor. 2 Lemma.

and in same manner it may be demonstrated, that *all the pls. which contain the solid polyhedron fall without the lesser sphere.*

∴ In the greater of two spheres which have same centre, a solid polyhedron is described, the superficies of which does not meet the lesser sphere. Q. E. F.

Another

PROP. XVII. CONTINUED.

Another and shorter demonstration that AZ > AG without the aid of Prop. XVI.

From G, draw GU rt. ∠s to AG;
and join AU.

If then $\widehat{BE}$ be bisec. continually there will at length be left an arc > arc which is subtend. by a rt. line = GU inscribed in the ⊙BCDE;

let this be $\widehat{KB}$;

∴ KB	$<$	GU;	
and ∵ ∠ BZK	$>$	rt. ∠,	demon.
∴ BK	$>$	BZ:	
but GU	$>$	BK,	
much more than GU	$>$	BZ,	
and GU^2	$>$	BZ^2:	
and AU	$=$	AB,	
∴ AU^2, i. e. AG^2+GU^2	$=$	AB^2, i. e. AZ^2+ZB^2;	
but BZ^2	$<$	GU^2,	
∴ AZ^2	$>$	AG^2;	
and consequently AZ	$>$	AG.	

Q. E. D.

Cor. And if in the lesser sphere there be inscribed a solid polyhedron, by drawing right lines betwixt the points in which the right lines from the centre of the sphere drawn to all the angles of the solid polyhedron in the greater sphere meet the superficies of the lesser; in the same order in which are joined the points in which the same lines from the centre meet the superficies of the greater sphere; the solid polyhedron in the sphere BCDE shall have to this other solid polyhedron the triplicate ratio of that which the diameter of the sphere BCDE has to the diameter of the other sphere. For if these two solids be divided into the same number of pyramids, and in the same order, the pyramids shall be similar to each other, each

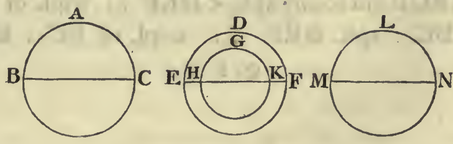
PROP. XVII. CONTINUED.

each to each: because they have the solid angles at their common vertex, the centre of the sphere, the same in each pyramid, and their other solid angles at the bases, equal to each other, each to each, because they are contained by three plane angles, each equal to each; and the pyramids are contained by the same number of similar planes; and are therefore similar to each other, each to each: but similar pyramids have to each other the triplicate ratio of their homologous sides: therefore the pyramid of which the base is the quadrilateral KBOS, and vertex A, has to the pyramid in the other sphere of the same order, the triplicate ratio of their homologous sides, that is, of that ratio which AB from the centre of the greater sphere has to the right line from the same centre to the superficies of the lesser sphere. And in like manner, each pyramid in the greater sphere has to each of the same order in the lesser, the triplicate ratio of that which AB has to the semi-diameter of the lesser sphere. And as one antecedent is to its consequent, so are all the antecedents to all the consequents. Wherefore the whole solid polyhedron in the greater sphere has to the whole solid polyhedron in the other, the triplicate ratio of that which AB the semi-diameter of the first has to the semi-diameter of the other; that is, which the diameter BD of the greater has to the diameter of the other sphere.

PROP. XVIII.—Theorem.

Spheres have to each other the triplicate ratio of that which their diameters have.

Let ABC, DEF be two spheres of which the diams. are BC, EF. The sphere ABC : sph. DEF : : tripl. of BC : EF.



For, if it have not,

then sph. ABC : $\left\{\begin{matrix}\text{a sph.} > \text{or} \\ < \text{DEF}\end{matrix}\right\}$: : tripl. of BC : EF.

First—Let it have this ratio to GHK < sph. DEF;

and let DEF have same cent. with GHK;

in greater sph. DEF descr. a sol. polyhed. whose pls. do not meet GHK;

and in sph. ABC descr. another polyhed. simil. that in DEF;

$\therefore \left\{\begin{matrix}\textit{sol. polyhed.} \\ \text{in sph. ABC}\end{matrix}\right\} : \left\{\begin{matrix}\textit{sol. polyhed.} \\ \text{in sph. DEF}\end{matrix}\right\}$: : tripl. of BC : EF. cor. 17. 11.

But sph. ABC : sph. GHK : : tripl. of BC : EF,

$\therefore \left.\begin{matrix}\text{sph. ABC :} \\ \text{sph. GHK}\end{matrix}\right\}$: : $\left\{\begin{matrix}\text{sol. polyhed.} \\ \text{in sph. ABC}\end{matrix}\right\} : \left\{\begin{matrix}\text{sol. polyhed.} \\ \text{in sph. DEF.}\end{matrix}\right.$

But sph. ABC > polyhed. inscr. in it,

∴ also sph. GHK > polyhed. in sph. DEF; 14. 5.

but also sph. GHK < polyhed. in sph. DEF,

for it is contained within it,

which is impossible:

∴ sph. ABC is not to any sph. < DEF : : tripl. of BC : EF;

similarly sph. DEF is not to any sph. < ABC : : tripl. of BC : EF.

Neither can sph. ABC : any sph. > DEF : : tripl. of BC : EF.

For

PROP. XVIII. CONTINUED.

For if possible,

Secondly—Let ABC : sph. LMN > DEF : : tripl. of BC : EF;
∴ invert. LMN : ABC : : tripl. of EF : BC.
But LMN : ABC : : DEF : a sph. < ABC, 14. 5.
(for LMN > DEF),
∴ DEF : some sph. < ABC : : tripl. of EF : BC;
which was demon. impossible;
∴ sph. ABC is not to any sph. > DEF : : tripl. of BC : EF;
also sph. ABC is not to any sph. < DEF : : tripl. of BC : EF;
∴ sph. ABC : sph. DEF : : tripl. of BC : EF.

Q. E. D.

FINIS.

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